OCR Further Pure 1 Module Revision Sheet

The FP1 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted. . . always an area prone to error.

J .M .S .

Summing Series

- You must know the important results:

$$\begin{aligned}\sum_{r=1}^{n} 1 &= 1+1+1+\cdots+1 = n,\\ \sum_{r=1}^{n} r &= 1+2+3+\cdots+n = \tfrac{1}{2}n(n+1),\\ \sum_{r=1}^{n} r^2 &= 1^2+2^2+3^2+\cdots+n^2 = \tfrac{1}{6}n(n+1)(2n+1),\\ \sum_{r=1}^{n} r^3 &= 1^3+2^3+3^3+\cdots+n^3 = \tfrac{1}{4}n^2(n+1)^2.\end{aligned}$$

- Also know the important properties (for constant λ and μ)

$$\sum_{r=1}^{n}(\lambda f(r) \pm \mu g(r)) = \lambda\sum_{r=1}^{n} f(r) \pm \mu\sum_{r=1}^{n} g(r).$$

  However beware of these!

$$\sum_{r=1}^{n}(f(r)\times g(r)) \neq \sum_{r=1}^{n} f(r)\times\sum_{r=1}^{n} g(r) \qquad \text{and} \qquad \sum_{r=1}^{n}\left(\frac{f(r)}{g(r)}\right) \neq \frac{\sum_{r=1}^{n} f(r)}{\sum_{r=1}^{n} g(r)}.$$

  To apply the first of these is equivalent to the heinous crime of $(a+b+c)^2 = a^2+b^2+c^2$!!!

- These must be applied in cases such as:

$$\begin{aligned}\sum_{r=1}^{n}(4r^2-2r+3) &= 4\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} 1\\ &= \tfrac{2}{3}n(n+1)(2n+1) - n(n+1) + 3n\\ &= \tfrac{n}{3}[2(n+1)(2n+1) - 3(n+1) + 9]\\ &= \tfrac{n}{3}(4n^2+3n+8).\end{aligned}$$

- If the sum starts from a number other than 1 then you can use the trick (which should be obvious)

$$\sum_{r=a}^{n}(\text{something}) = \sum_{r=1}^{n}(\text{something}) - \sum_{r=1}^{a-1}(\text{something}).$$

- The *method of differences* can be used to sum certain expressions where cancellation occurs when the sum is written out. For example find $\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$. Write the sum out, starting a new line for each value of r and you should see that some nice cancelling occurs;

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) &= \frac{1}{3}-\frac{1}{5}\\ &+\frac{1}{5}-\frac{1}{7}\\ &+\frac{1}{7}-\frac{1}{9}\\ &\quad\vdots\\ &+\frac{1}{2n+1}-\frac{1}{2n+3}.\end{aligned}$$

  You can see that everything cancels except the $\frac{1}{3}$ and the $\frac{1}{2n+3}$ so

$$\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)=\frac{1}{3}-\frac{1}{2n+3}.$$

  It is usually best *not* to combine these terms together into one fraction in order to make it easier to see if there is a sum to infinity.

- A sum to infinity exists if the expression for the sum to n has a finite limit as $n\to\infty$. In the above example it does, so

$$\sum_{r=1}^{\infty}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n+3}\right)=\frac{1}{3}.$$

- Questions of this sort invariably start "Show that $f(r)-g(r)=h(r)$", and then ask you to sum $h(r)$; this, clearly, is the same as summing $f(r)-g(r)\Rightarrow$ 'method of differences'.

Matrices

- Capital letters tend to be used to denote matrices and you should underline them, just as you do with vectors. An $n\times m$ matrix has n rows and m columns. So $\begin{pmatrix}1 & 2 & -3\\ 2 & -2 & 7\end{pmatrix}$ is a 2×3 matrix. You must be able to add, subtract and multiply matrices. To add or subtract matrices they must be the same size and it works as you would expect. To multiply matrices ($\mathbf{A}\times\mathbf{B}$, say) the number of columns of $\mathbf{A}$ must be the same as the number of rows of $\mathbf{B}$. Your teacher will have explained this better than I ever can here, but a few examples: test for yourself!

$$\begin{pmatrix}1 & 5\\ -2 & 1\end{pmatrix}\begin{pmatrix}2 & 7 & -1\\ 0 & 1 & 4\end{pmatrix}=\begin{pmatrix}2 & 12 & 19\\ -4 & -13 & 6\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 4\end{pmatrix}=\begin{pmatrix}-2\end{pmatrix},$$

$$\begin{pmatrix}1\\ 0\\ 4\end{pmatrix}\begin{pmatrix}2 & 3 & -1\end{pmatrix}=\begin{pmatrix}2 & 3 & -1\\ 0 & 0 & 0\\ 8 & 12 & -4\end{pmatrix}.$$

- The determinant of an $n\times n$ ('*square*') matrix can be denoted by the letter Δ. A matrix with $\Delta=0$ is called a '*singular*' matrix; otherwise it is '*non-singular*'. For a 2×2 matrix $\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}\equiv\Delta=ad-bc$.

- The inverse of a 2×2 matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The inverse of a matrix (if it exists) is such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- Matrix multiplication is not, in general, commutative; i.e. $\mathbf{AB} \neq \mathbf{BA}$. Matrix multiplication is, however, associative; i.e. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An extension of this is $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \dots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \dots \mathbf{A}_3^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$; prove it by induction yourself if you fancy...

- You must be very careful when manipulating matrix equations because of this non-commutativity. With normal numbers we are happy with $ax = b$ giving $x = ba^{-1}$, but this is wrong in matrix-world.

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \quad \underline{\text{pre}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B}. \end{aligned}$$

Or

$$\begin{aligned} \mathbf{XA} &= \mathbf{B} \\ \mathbf{XAA}^{-1} &= \mathbf{BA}^{-1} \quad \underline{\text{post}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{XI} &= \mathbf{BA}^{-1} \\ \mathbf{X} &= \mathbf{BA}^{-1}. \end{aligned}$$

- Know that linear simultaneous equations can be expressed by matrices:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}.$$

The system of equations has a unique solution provided $\Delta \neq 0$. If $\Delta = 0$ then there are no unique solutions: there are either an infinite set of solutions or no solutions at all depending on whether $ax + by = c$ and $dx + ey = f$ represent parallel lines (no solutions) or the same line (infinite set of solutions).

The unique solution (if it exists) is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ae-bd}\begin{pmatrix} e & -b \\ -d & a \end{pmatrix}\begin{pmatrix} c \\ f \end{pmatrix}$.

Matrix Transformations

- Matrices can be thought of as transformations. To discover what a matrix does, consider what it does to the arbitrary point (x, y) and *think*! For example

  1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ so $(x, y) \to (x, y)$. Therefore matrix does nothing.
  2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ so $(x, y) \to (y, x)$. Therefore the x and y-coordinates get flipped, so matrix reflected in the line $y = x$.

3. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ so $(x, y) \to (-x, y)$. Therefore matrix changes the sign of the x-coordinate, so it represents a reflection in the y-axis.
4. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$ so $(x, y) \to (y, -x)$. Draw a few sample points and we see it represents a rotation 90° clockwise about the origin.
5. $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$ so $(x, y) \to (4x, 4y)$. So the x and y-coordinates get multiplied by 4. Therefore an enlargement scale factor 4, centre the origin.

- You need to know the family of matrices that represent shears.

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \text{“Shear with } x\text{-axis invariant with shear constant } k\text{”}. \quad \rightleftarrows$$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \text{“Shear with } y\text{-axis invariant with shear constant } k\text{”}. \quad \downarrow\uparrow$$

- For combined transformations you write the matrices in the opposite order to which the transformations occur[1]. For example if we apply transformation **A** followed by transformation **B**, then the matrix for this combined transformation would be **BA**.

- If a 2×2 matrix $\mathbf{M}$ represents a transformation, then $|\det(\mathbf{M})|$ represents the *area scale factor* of the transformation.

  For example $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement with length scale factor 2. We see the determinant is 4, so areas get multiplied by 4 in the transformation, which is consistent.

- If we consider an arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the point $(1, 0)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

  (i.e. the first column of the matrix). Similarly if we act on the point $(0, 1)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

  (i.e. the second column of the matrix). This immensely powerful pair of statements tells us that *if* a transformation can be expressed by a matrix, then all we need to do to find the matrix that does what we want is to find where $(1, 0)$ maps to under the transformation and write this image point as the first column of our matrix and find where $(0, 1)$ maps to under the transformation and write this as the second column.

- For example find the matrix that:

  – reflects in the x-axis. $(1, 0) \to (1, 0)$ and $(0, 1) \to (0, -1)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

  – reflects in the $y = x$. $(1, 0) \to (0, 1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

  – rotates 90° clockwise. $(1, 0) \to (0, -1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[1] Just like functions: If you apply f then g, we do $gf(x)$.

– enlarges scale factor 3. $(1,0) \to (3,0)$ and $(0,1) \to (0,3)$ $\Rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

– stretch factor 2 parallel to y-axis. $(1,0) \to (1,0)$ and $(0,1) \to (0,2)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

– rotates $\theta°$ anticlockwise. $(1,0) \to (\cos\theta, \sin\theta)$ and $(0,1) \to (-\sin\theta, \cos\theta)$
$\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

– rotates 90° CW and then reflects in y-axis. $(1,0) \to (0,-1)$ and $(0,1) \to (1,0)$
$\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3 × 3 Matrices

- To calculate the determinant of a 3×3 matrix, you pick a column or a row (most students choose the first column, but it works with any row or column) and you work down/across it using the plus/minus checkerboard approach and multiplying by the determinant of the 2×2 matrix left when the column and row of the number you have chosen is crossed out.

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \Delta = a(ei - fh) - d(bi - ch) + g(bf - ce).$$

- To invert a 3×3 matrix you do:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} +(ei-hf) & -(di-gf) & +(dh-eg) \\ -(bi-ch) & +(ai-cg) & -(ah-bg) \\ +(bf-ce) & -(af-cd) & +(ae-bd) \end{pmatrix}^{\mathrm{T}},$$
$$= \frac{1}{\Delta} \begin{pmatrix} +(ei-hf) & -(bi-ch) & +(bf-ce) \\ -(di-gf) & +(ai-cg) & -(af-cd) \\ +(dh-eg) & -(ah-bg) & +(ae-bd) \end{pmatrix}.$$

  Don't forget to transpose at the end! There is an elegant pattern to all of the above; it's easy to do once you get into the swing of it. (Mr Stone has a spreadsheet where you can practice this to your heart's content.)

- As before, a system of linear simultaneous equations can be written with a matrix.

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d \\ h \\ l \end{pmatrix}.$$

  If the matrix in non-singular then the equations have a unique solution. If the matrix is singular then the system either has an infinite set of solutions or no solutions at all. If the equations generate an inconsistency (e.g. 4=19) then there are no solutions at all.

- If the matrix is non-singular, then the unique solution is given by:

$$\mathbf{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Complex Numbers

- Complex numbers start with one idea only; that we can find a number that squares to -1; we call it i. Therefore $i^2 = -1$. It is not a number that exists on the number line so it is refered to as *complex* or *imaginary*. Therefore the square root of any negative number can now be calculated; $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$.

- In general a complex number can consist of a real part and an imaginary/complex part i.e. $a+ib$, where a is the real part and b is the complex part. We write $\text{Re}(a+ib) = a$ and $\text{Im}(a+ib) = b$. It is important to note that a and b themselves *must* be real numbers.

- We can use complex numbers to solve *any* quadratic equation. For example solve $3x^2 + 2x + 7 = 0$ by the formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4\times 3\times 7}}{2\times 3} = -\frac{1}{3} \pm i\frac{2\sqrt{5}}{3}.$$

- A complex (or real) number can be represented as a point in an *Argand* diagram. So the complex number $6+2i$ would be the point 6 across and 2 up, at the equivalent point where $(6,2)$ would be in a Cartesian coordinate system.

- The complex conjugate ($z*$) of a complex number (z) is where the complex part has the sign changed. For example if $z = 3-7i$, then $z* = 3+7i$. Real numbers are, therefore, their own conjugates. Any number with an imaginary component is reflected in the real axis in the Argand diagram.

- If two complex numbers are equal, then the real parts must be equal and the complex parts must be equal: i.e.

$$\begin{aligned} z_1 = z_2 \quad &\Rightarrow \quad \text{Re}(z_1) = \text{Re}(z_2) \quad \text{and} \quad \text{Im}(z_1) = \text{Im}(z_2), \\ a+ib = c+id \quad &\Rightarrow \quad a = c \quad \text{and} \quad b = d. \end{aligned}$$

- To add, subtract or multiply complex numbers the results are pretty obvious:

$$\begin{aligned} (a+bi)+(c+id) &= (a+c)+i(b+d) \\ (a+bi)-(c+id) &= (a-c)+i(b-d) \\ (a+bi)(c+id) &= ac+adi+bci+bdi^2 = (ac-bd)+i(ad+bc) \end{aligned}$$

- To divide by a complex number use a trick taken from surds; in C1 if the bottom line was $a \pm b\sqrt{k}$ then you multiplied top and bottom by $a \mp b\sqrt{k}$. In FP1 if you want to divide by $a \pm ib$, then you multiply top and bottom by the complex conjugate $a \mp ib$. For example:

$$\frac{3-2i}{2+5i} = \frac{3-2i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{6+10i^2-4i-15i}{4-25i^2+10i-10i} = \frac{-4-19i}{29}.$$

- Complex numbers exhibit the elegant property of *closure*[2]. This means that any operation on complex numbers involving $+, -, \times, \div, \sqrt{\ldots}, \sqrt[n]{\ldots}$ etc. will produce an answer that is also complex[3]. This allows us to state that the answer to a given problem *must* be $a+ib$ for some a and b and then proceed to calculate a and b by equating the real part and, separately, the imaginary part.

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[2] "And that, my friend, is what they call closure" - *Rachel Green, Friends*

[3] Note this does not happen with the real numbers; you cannot always square root a number.

- In the above problem to find $\frac{3-2i}{2+5i}$ we could also have approached it by stating that the answer is $a+ib$ and manipulating:

$$\begin{aligned}\frac{3-2i}{2+5i} &= a+ib\\ 3-2i &= (a+ib)(2+5i)\\ 3-2i &= (2a-5b)+i(5a+2b).\end{aligned}$$

This yields the simultaneous equations $3 = 2a-5b$ and $-2 = 5a+2b$. These solve to $a = -\frac{4}{29}$ and $b = -\frac{19}{29} \Rightarrow \frac{-4-19i}{29}$, just as before. I wouldn't use this method in this case but I would certainly use it...

- ...to find square roots. The square roots of 16 are (obviously) ± 4. With the exception of zero, we should expect two roots and the same is true of complex numbers. For example find the square roots of $8-6i$: We know that the answers must be of the form $a+ib$ such that

$$\begin{aligned}8-6i &= (a+ib)^2\\ 8-6i &= (a^2-b^2)+(2ab)i\\ \text{Therefore, } 8 &= a^2-b^2 \text{ and } -6 = 2ab.\end{aligned}$$

From the second we find $b = -\frac{3}{a}$. Putting this in the first we find $0 = a^4 - 8a^2 - 9 = (a^2-9)(a^2+1)$. The first bracket yields $a = \pm 3$. (The second bracket yields $a = \pm i$, but we can discard this because a must be real.) Therefore this yields the square roots $3-i$ and $-3+i$. In the Argand diagram you should find that square roots come out in opposite directions from the origin.

- If a polynomial has *real coefficients* then its roots are either real, or exist in complex conjugate pairs. Therefore if $z = a \pm ib$ is a root, then so is $z = a \mp ib$.

  For example, given that $z^4 - z^3 + 2z^2 + 7z - 5 = 0$ has one root $1-2i$, solve the equation fully. Since the coefficients are real we know that the conjugate $1+2i$ must also be a root. Therefore $(z-(1-2i))$ and $(z-(1+2i))$ must be factors by the factor theorem. Multiplying out the two factors we find $(z-(1-2i))(z-(1+2i)) = (z^2-2z+5)$ which must also be a factor. By polynomial division we find

$$z^4 - z^3 + 2z^2 + 7z - 5 = (z^2-2z+5)(z^2+z-1) = 0.$$

The second quadratic solves to $z = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Therefore the solutions are

$$z = 1-2i, \quad z = 1+2i, \quad z = -\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}, \quad z = -\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}.$$

- The modulus of a complex number ($z = x+iy$) is defined $|z| = \sqrt{x^2+y^2}$. It represents the distance of a complex number from the origin. For example the modulus of $z = 2-2\sqrt{3}i$ would be $|z| = \sqrt{2^2+(-2\sqrt{3})^2} = 4$.

- The argument of a complex number is defined as the angle a line from the origin to a complex number makes with the positive real axis. By convention $-\pi < \arg(z) \leqslant \pi$. For example

$$\arg(4) = 0,$$
$$\arg(i) = \frac{\pi}{2},$$
$$\arg(-3) = \pi,$$
$$\arg(1+i) = \frac{\pi}{4},$$
$$\arg(-1-i) = -\frac{3\pi}{4}.$$

Arguments are best calculated by drawing a suitable right angled triangle in an Argand diagram and then calculating the desired angle (not always an angle in the triangle you've drawn, but π minus it, etc.)

- You must be able to sketch loci of points obeying a rule defined by a modulus or an argument. The most important fact here is often that the operation *subtraction* takes you *from* one complex number *to* another[4]; i.e. $z - w$ takes you *from* w *to* z. An addition can be converted into a subtraction by $z + w = z - (-w)$; this therefore represents the movement from $-w$ to z.
- This idea allows us to draw certain loci very easily indeed.

  For example: $|z| = 4$ means the length of z from the origin is 4; i.e. a circle of radius 4, centre the origin.

  For example: $|z - 2| = 5$ means the length travelling from 2 to z is 5, so a circle radius 5, centre 2(+0i).

  For example: $|z + i| < 2$ is the same as $|z - (-i)| < 2$ which means the length travelling from $-i$ to z is less than 2, so the inside of a circle radius 2, centre $(0) - i$.

  For example: $|z| = |z + 1 - i|$ is the same as $|z| = |z - (-1 + i)|$ which means the length travelling from 0 to z must be the same as the distance travelling from $-1 + i$ to z so it must be the perpendicular bisector of 0 and $-1 + i$, i.e. $y = x + 1$.
- The above type of questions can also be done using a method in your textbook (see top of P144, "Method 2"), but I prefer the 'intuitive' way demonstrated above.
- Argument loci also come up and we can use the same principles.

  For example: $\arg(z) = \frac{\pi}{2}$ means the argument z makes is $\frac{\pi}{2}$ so it is a vertical line going up from the origin (with a hollow circle drawn at the origin to indicate that it is not included in the half-line).

  For example: $\arg(z - i) = \frac{\pi}{6}$ means the argument going from i to z is $\frac{\pi}{6}$ so it is a half-line from i (hollow circle) at angle $\frac{\pi}{6}$ with the positive real axis.

Roots Of Equations

- By considering the general quadratic equation $ax^2 + bx + c = 0$ we re-write it as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Quadratics can be factorised into two linear factors $(x - \alpha)(x - \beta)$. By equating the two we find

$$(x-\alpha)(x-\beta) = x^2 + \frac{b}{a}x + \frac{c}{a}$$
$$x^2 - (\alpha+\beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}.$$

[4] In precisely the same way that with vectors $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

So we see that the sum of the roots of a quadratic is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$.

- By two tedious derivations (that you *should* do for yourself) similar to the one above we find that for the cubic ($ax^3+bx^2+cx+d=0$) and the quartic ($ax^4+bx^3+cx^2+dx+e=0$) the following:

Quadratics	Cubics	Quartics
$\alpha+\beta=-\frac{b}{a}$	$\alpha+\beta+\gamma=-\frac{b}{a}$	$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$
$\alpha\beta=\frac{c}{a},$	$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$
	$\alpha\beta\gamma=-\frac{d}{a},$	$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$
		$\alpha\beta\gamma\delta=\frac{e}{a}.$

For your exam you only need quadratics and cubics, but the pattern continues fairly easily to quartics, quintics and beyond.

- For speed of writing we use the following shorthand:

$$\alpha+\beta+\gamma\equiv\sum\alpha \qquad \text{and} \qquad \alpha\beta+\alpha\gamma+\beta\gamma\equiv\sum\alpha\beta.$$

- We can therefore find properties of roots from equations without having to solve the equations themselves.

  For example from $2x^2-3x-6=0$ I can say that $\alpha\beta=\frac{-6}{2}=-3$ and $\alpha+\beta=\frac{3}{2}$.

  For example from $2x^3-4x^2-3x+6=0$ I can say that $\alpha\beta\gamma=\frac{-6}{2}=-3$, $\sum\alpha\beta=-\frac{3}{2}$ and $\sum\alpha=\frac{4}{2}=2$. Watch those signs!

- You are often asked to construct new equations with roots related to the original equation's roots. There are two basic methods for this:

- Method I (the "Lo-Tech" approach) is to find the new 'sum'/'sum of prod'/'prod' of roots etc., from knowledge of the old 'sum'/'sum of prod'/'prod'. Two examples:

  1. The equation $2x^2+5x+7=0$ has roots α and β. Find an equation with roots $2\alpha-1$ and $2\beta-1$. We can see that $\alpha\beta=\frac{7}{2}$ and $\alpha+\beta=-\frac{5}{2}$. Therefore for the new equation must have the following:
     New sum of roots $=(2\alpha-1)+(2\beta-1)=2(\alpha+\beta)-2=2\times(-\frac{5}{2})-2=-7$.
     New prod of roots $=(2\alpha-1)(2\beta-1)=4\alpha\beta-2(\alpha+\beta)+1=4\times\frac{7}{2}-2\times(-\frac{5}{2})+1=20$.
     Therefore the new equation is $u^2+7u+20=0$.
  2. The equation $2x^3-x^2+4x+2=0$ has roots α, β and γ. Find an equation with roots $\alpha+1$, $\beta+1$ and $\gamma+1$. We can see that $\alpha\beta\gamma=-1$, $\sum\alpha\beta=2$ and $\sum\alpha=\frac{1}{2}$. Therefore for the new equation we must have the following:
     New sum of roots $=(\alpha+1)+(\beta+1)+(\gamma+1)=\sum\alpha+3=\frac{7}{2}$.
     New sum of prods $=(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=\sum\alpha\beta+2\sum\alpha+3=2+2\times\frac{1}{2}+3=6$.
     New prod of roots $=(\alpha+1)(\beta+1)(\gamma+1)=\alpha\beta\gamma+\sum\alpha\beta+\sum\alpha+1=-1+2+\frac{1}{2}+1=\frac{5}{2}$.
     Therefore the equations becomes $u^3-\frac{7}{2}u^2+6u-\frac{5}{2}=0$ which we double to make it 'nice':
     $$2u^3-7u^2+12u-5=0.$$

- Method II (the "Hi-Tech" approach) is to make a substitution into the original equation to construct a second. We will do the same two examples as above.

  1. The equation $2x^2+5x+7=0$ has roots α and β. Find an equation with roots $2\alpha-1$ and $2\beta-1$. Let u be one of the new roots; $u=2\alpha-1$. Rearrange to make α the subject; $\alpha=\frac{u+1}{2}$. We know that α satisfies the original equation because it is a root, so if we substitute $\alpha=\frac{u+1}{2}$ into the original equation we will have an equation in u which has the desired roots.

  $$2\left(\frac{u+1}{2}\right)^2+5\left(\frac{u+1}{2}\right)+7=0 \quad\Rightarrow\quad u^2+7u+20=0.$$

  2. The equation $2x^3-x^2+4x+2=0$ has roots α, β and γ. Find an equation with roots $\alpha+1$, $\beta+1$ and $\gamma+1$. So, let $u=\alpha+1$. Therefore $\alpha=u-1$. Sub in we find:

  $$2(u-1)^3-(u-1)^2+4(u-1)+2=0 \quad\Rightarrow\quad 2u^3-7u^2+12u-5=0.$$

Proof by Induction

- Let $P(n)$ be a proposition which depends on some integer value n. The principle of induction works as follows: Start by demonstrating the truth of $P(1)$ (say). Then we show that <u>*if*</u> $P(k)$ is true for *some value* k <u>*then*</u> it implies the truth of $P(k+1)$, then $P(n)$ must be true for all integer $n\geqslant 1$. This is because we have shown

  $$P(1)\Rightarrow P(2),\text{ and } P(2)\Rightarrow P(3),\text{ and } P(3)\Rightarrow P(4)\text{ etc. etc. etc.}$$

- Your answer should always follow this template:
  - "Let $P(n)$ be the proposition that $f(n)=g(n)$ for all $n\geqslant 1$."
  - "Basis Case: If $n=1$, $f(1)=\ldots$ and $g(1)=\ldots$. We see $f(1)=g(1)$ so $P(1)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n=k$:

    $$f(k)=g(k)."$$

    [Then manipulate $f(k)=g(k)$ using algebra to obtain the next line]

    $$"f(k+1)=g(k+1)."$$
  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that *if* $P(k)$ is true *then* $P(k+1)$ is also true and since $P(1)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n\geqslant 1$."

- If an induction question includes a "$\sum$", can I suggest you get rid of it by writing out the sum term-by-term; students tend to get muddled on when to use r, n, k and $k+1$ in my experience (although this might be my teaching). Also leave initial numerical values unevaluated; 1×2^2 is preferable to 4. For example

  $$\sum_{r=1}^{n} r(r+2)\Rightarrow 1\times 3+2\times 4+\cdots+n(n+2).$$

- For example use induction to prove that for $n \geqslant 2$, $\sum_{r=2}^{n}(r-1)r = \frac{1}{3}n(n-1)(n+1)$.
  - "The question is the same as proving $1\times 2+2\times 3+\cdots+(n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Let $P(n)$ be the proposition $1\times 2+2\times 3+\cdots+(n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Basis case: If $n=2$, $1\times 2+2\times 3+\cdots+(n-1)n = 2$ and $\frac{1}{3}n(n-1)(n+1) = 2$. We see that LHS = RHS = 2 so $P(2)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n=k$:
  $$1\times 2+2\times 3+\cdots+(k-1)k = \frac{1}{3}k(k-1)(k+1)."$$
  - (Add the next term to the LHS to both sides:)
  $$\begin{aligned}``1\times 2+2\times 3+\cdots+(k-1)k+\underline{k(k+1)} &= \frac{1}{3}k(k-1)(k+1)+\underline{k(k+1)}\\ &= \frac{1}{3}k(k+1)[(k-1)+3]\\ &= \frac{1}{3}k(k+1)(k+2)."\end{aligned}$$
  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that if $P(k)$ is true then $P(k+1)$ is also true and since $P(2)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \geqslant 2$."
- In a recent official mark scheme, the use of the words 'mathematical induction' in your conclusion was needed for full marks.