

Pure
Further Mathematics 3

Revision Notes

June 2016

1	Hyperbolic functions	3
	Definitions and graphs.....	3
	Addition formulae, double angle formulae etc.	3
	Osborne's rule	3
	Inverse hyperbolic functions	4
	Graphs	4
	Logarithmic form.....	4
	Equations involving hyperbolic functions.....	5
2	Further coordinate systems	6
	Ellipse.....	6
	Hyperbola.....	7
	Parabola.....	7
	Parametric differentiation.....	8
	Tangents and normals.....	8
	Finding a locus	9
3	Differentiation	10
	Derivatives of hyperbolic functions	10
	Derivatives of inverse functions.....	10
4	Integration	12
	Standard techniques.....	12
	Recognise a standard function	12
	Using formulae to change the integrand	12
	Reverse chain rule	12
	Standard substitutions.....	13
	Integration inverse functions and $\ln x$	14
	Reduction formulae	15
	Arc length.....	18
	Area of a surface of revolution.....	20
5	Vectors	22
	Vector product.....	22
	The vectors i, j and k	22
	Properties.....	22
	Component form	23
	Applications of the vector product	23
	Volume of a parallelepiped	24
	Triple scalar product.....	24
	Volume of a tetrahedron.....	25
	Equations of straight lines	25
	Vector equation of a line.....	25
	Cartesian equation of a line in 3-D	26
	Vector product equation of a line.....	26
	Equation of a plane.....	27
	Scalar product form	27
	Cartesian form	27
	Vector equation of a plane.....	28
	Distance from a point to a plane	29
	Distance from any point to a plane	29

Reflection of a point in a plane	30
Distance between parallel planes	31
Shortest distance from a point to a line	31
Projections – an alternative approach	32
Shortest distance from a point from a plane.....	32
Distance between parallel planes	33
Shortest distance between two skew lines.....	34
Shortest distance from a point to a line	35
Line of intersection of two planes	36
Angle between line and plane	37
Angle between two planes	38
6 Matrices	39
Basic definitions.....	39
Dimension of a matrix	39
Transpose and symmetric matrices	39
Identity and zero matrices	39
Determinant of a 3×3 matrix	39
Properties of the determinant	40
Singular and non-singular matrices.....	40
Inverse of a 3×3 matrix	40
Cofactors.....	40
Finding the inverse.....	41
Properties of the inverse.....	41
Matrices and linear transformations.....	42
Linear transformations	42
Base vectors i, j, k	42
Image of a line	44
Image of a plane 1	44
Image of a plane 2.....	44
Method 1	44
Method 2, as in the book.....	45
7 Eigenvalues and eigenvectors.....	46
Definitions	46
2×2 matrices.....	46
Orthogonal matrices.....	47
Normalised eigenvectors.....	47
Orthogonal vectors.....	47
Orthogonal matrices.....	47
Diagonalising a 2×2 matrix	48
Diagonalising 2×2 symmetric matrices.....	49
Eigenvectors of symmetric matrices	49
Diagonalising a symmetric matrix	49
3×3 matrices.....	51
Finding eigenvectors for 3×3 matrices.....	51
Diagonalising 3×3 symmetric matrices.....	52

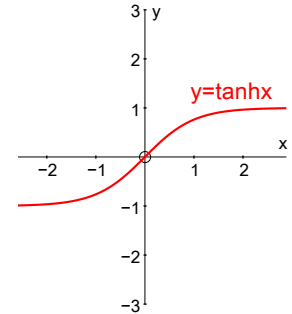
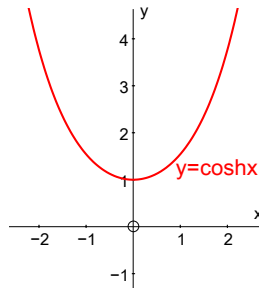
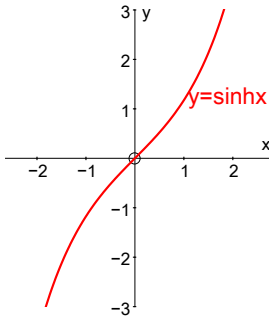
1 Hyperbolic functions

Definitions and graphs

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

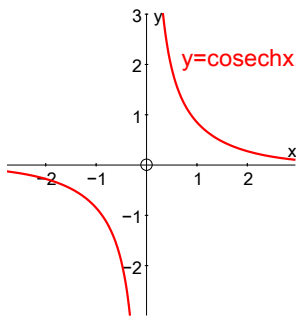
$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

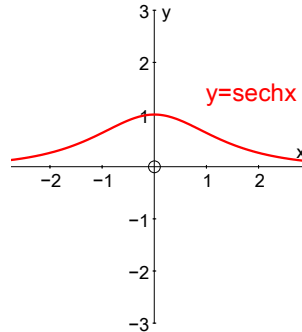


You should be able to draw the graphs of $\operatorname{cosech} x$, $\operatorname{sech} x$ and $\operatorname{coth} x$ from the above:

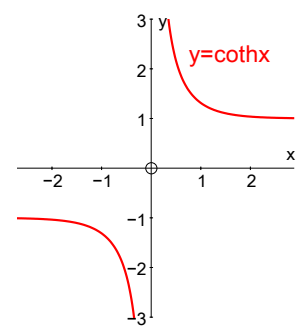
cosech x



sech x



coth x



Addition formulae, double angle formulae etc.

The standard trigonometric formulae are very similar to the hyperbolic formulae.

Osborne's rule

If a trigonometric identity involves the **product of two sines**, then we change the sign to write down the corresponding hyperbolic identity.

Examples:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\Rightarrow \sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$$

no change

but $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\Rightarrow \cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$$

product of two sines, so change sign

and $1 + \tan^2 A = \sec^2 A$

$$\Rightarrow 1 - \tanh^2 A = \operatorname{sech}^2 A$$

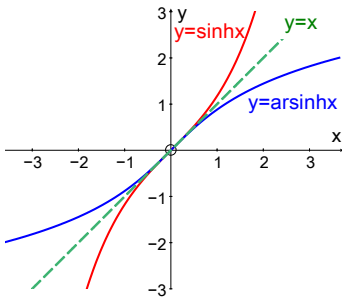
$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A}, \text{ product of two sines, so change sign}$$

Inverse hyperbolic functions

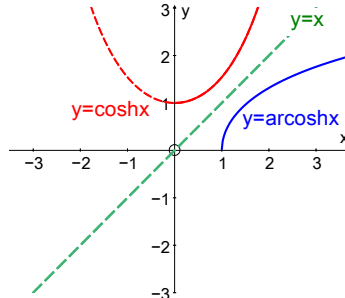
Graphs

Remember that the graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$.

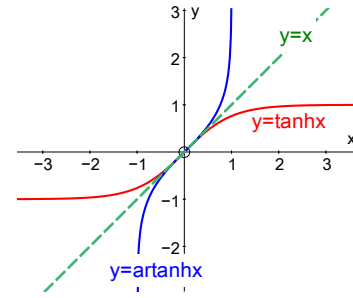
$$y = \operatorname{arsinh} x$$



$$y = \operatorname{arcosh} x$$



$$y = \operatorname{artanh} x$$



Notice $\operatorname{arcosh} x$ is a function defined so that $\operatorname{arcosh} x \geq 0$.

\Rightarrow there is only **one** value of $\operatorname{arcosh} x$.

However, the equation $\cosh z = 2$, has **two** solutions, $+\operatorname{arcosh} 2$ and $-\operatorname{arcosh} 2$.

Logarithmic form

1) $y = \operatorname{arsinh} x$

$$\Rightarrow \sinh y = \frac{1}{2}(e^y - e^{-y}) = x$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1} \quad \text{or} \quad x - \sqrt{x^2 + 1}$$

But $e^y > 0$ and $x - \sqrt{x^2 + 1} < 0 \Rightarrow e^y = x + \sqrt{x^2 + 1}$ **only**

$$\Rightarrow y = \operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$$

2) $y = \operatorname{arcosh} x$

$$\Rightarrow \cosh y = \frac{1}{2}(e^y + e^{-y}) = x$$

$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} \quad \text{both roots are positive}$$

$$\Rightarrow y = \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{or} \quad \ln(x - \sqrt{x^2 - 1})$$

It can be shown that $\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$

$$\Rightarrow y = \operatorname{arcosh} x = \pm \ln(x + \sqrt{x^2 - 1})$$

But $\operatorname{arcosh} x$ is a function and therefore has only one value (positive)

$$\Rightarrow y = \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

3) Similarly $\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

Equations involving hyperbolic functions

It would be possible to solve $6 \sinh x - 2 \cosh x = 7$ using the $R \sinh(x - \alpha)$ technique from trigonometry, but it is easier to use the exponential form.

Example: Solve $6 \sinh x - 2 \cosh x = 7$

Solution: $6 \sinh x - 2 \cosh x = 7$

$$\Rightarrow 6 \times \frac{1}{2}(e^x - e^{-x}) - 2 \times \frac{1}{2}(e^x + e^{-x}) = 7$$

$$\Rightarrow 2e^{2x} - 7e^x - 4 = 0$$

$$\Rightarrow (2e^x + 1)(e^x - 4) = 0$$

$$\Rightarrow e^x = -\frac{1}{2} \text{ (not possible) or } 4$$

$$\Rightarrow x = \ln 4$$

In other cases, the ‘trigonometric’ solution may be preferable

Example: Solve $\cosh 2x + 5 \sinh x - 4 = 0$

Solution: $\cosh 2x + 5 \sinh x - 4 = 0$

$$\Rightarrow 1 + 2 \sinh^2 x + 5 \sinh x - 4 = 0 \quad \text{note use of Osborn's rule}$$

$$\Rightarrow 2 \sinh^2 x + 5 \sinh x - 3 = 0$$

$$\Rightarrow (2 \sinh x - 1)(\sinh x + 3) = 0$$

$$\Rightarrow \sinh x = \frac{1}{2} \text{ or } -3$$

$$\Rightarrow x = \operatorname{arsinh} 0.5 \text{ or } \operatorname{arsinh} (-3)$$

$$\Rightarrow x = \ln(0.5 + \sqrt{0.5^2 + 1}) \text{ or } \ln((-3) + \sqrt{(-3)^2 + 1}) \quad \text{using log form of inverse}$$

$$\Rightarrow x = \ln\left(\frac{1+\sqrt{5}}{2}\right) \text{ or } \ln(\sqrt{10} - 3)$$

2 Further coordinate systems

Ellipse

Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

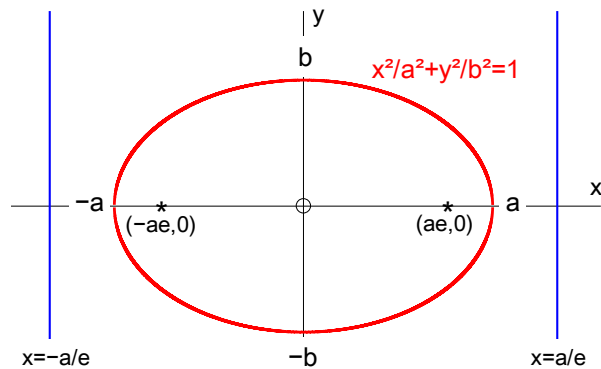
Parametric equations

$$x = a \cos \theta, \quad y = b \sin \theta$$

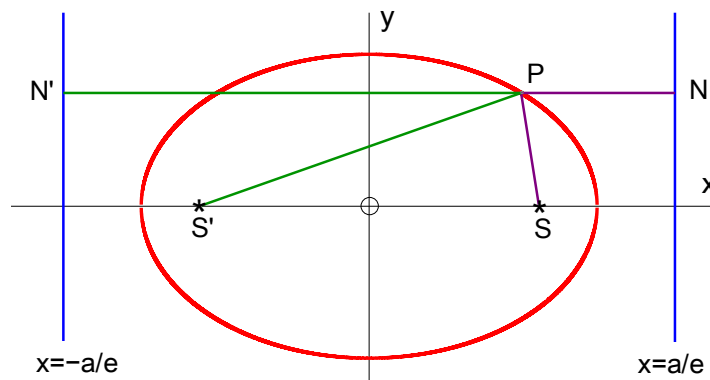
Foci at $S(ae, 0)$ and $S'(-ae, 0)$

Directrices at $x = \pm \frac{a}{e}$

Eccentricity $e < 1$, $b^2 = a^2(1 - e^2)$



An ellipse can be defined as the locus of a point P which moves so that $PS = e PN$, where S is one of the foci, $e < 1$ and N lies on the corresponding directrix.



This is true for either focus with the corresponding directrix.

$$\Rightarrow PS = e PN \text{ and } PS' = e PN'$$

$$\Rightarrow PS + PS' = e(PN + PN') = eNN'$$

$$\Rightarrow PS + PS' = e \frac{2a}{e} = 2a.$$

This justifies the 'string method' of drawing an ellipse.

Hyperbola

Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric equations

$$x = a \cosh \theta, \quad y = b \sinh \theta$$

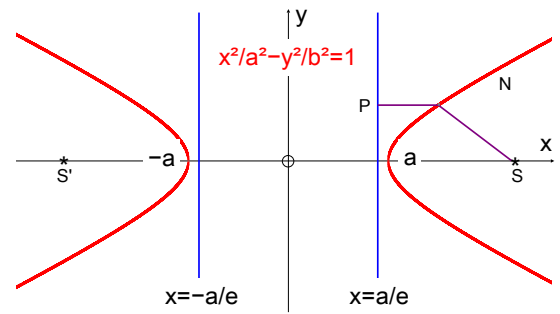
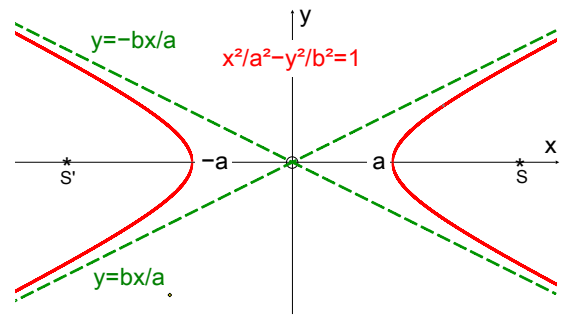
($x = a \sec \theta, \quad y = b \tan \theta$ also work)

Asymptotes $\frac{x}{a} = \pm \frac{y}{b}$

Foci at $S (ae, 0)$ and $S' (-ae, 0)$

Directrices at $x = \pm \frac{a}{e}$

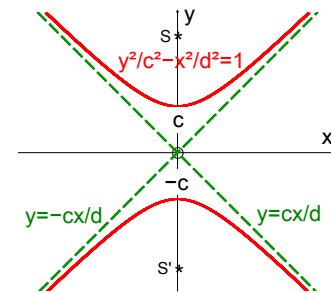
Eccentricity $e > 1, \quad b^2 = a^2(e^2 - 1)$



A hyperbola can be defined as the locus of a point P which moves so that $PS = e PN$, where S is the focus, $e > 1$ and N lies on the directrix.

$$\frac{y^2}{c^2} - \frac{x^2}{d^2} = 1$$

is a hyperbola with foci on the y -axis,



Parabola

Cartesian equation

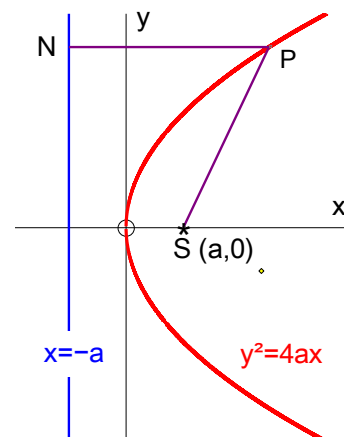
$$y^2 = 4ax$$

Parametric equations

$$x = at^2, \quad y = 2at$$

Focus at $S (a, 0)$

Directrix at $x = -a$



A parabola can be defined as the locus of a point P which moves so that $PS = PN$, where S is the focus, N lies on the directrix and the eccentricity $e = 1$.

Parametric differentiation

From the chain rule $\frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta}$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \text{or} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{using any parameter.}$$

Tangents and normals

It is now easy to find tangents and normals.

Example: Find the equation of the normal to the curve given by the parametric equations $x = 5 \cos \theta$, $y = 8 \sin \theta$ at the point where $\theta = \frac{\pi}{3}$

Solution: When $\theta = \frac{\pi}{3}$, $\cos \theta = \frac{1}{2}$ and $\sin \theta = \frac{\sqrt{3}}{2}$

$$\Rightarrow x = \frac{5}{2}, \quad y = 4\sqrt{3}$$

$$\text{and } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{8 \cos \theta}{-5 \sin \theta} = \frac{-8}{5\sqrt{3}} \quad \text{when } \theta = \frac{\pi}{3}$$

$$\Rightarrow \text{gradient of normal is } \frac{5\sqrt{3}}{8}$$

$$\Rightarrow \text{equation of normal is } y - 4\sqrt{3} = \frac{5\sqrt{3}}{8} \left(x - \frac{5}{2} \right)$$

$$\Rightarrow 5\sqrt{3}x - 8y + \frac{39\sqrt{3}}{2} = 0$$

Sometimes normal, or implicit, differentiation is (slightly) easier.

Example: Find the equation of the tangent to $xy = 36$, or $x = 6t$, $y = \frac{6}{t}$, at the point where $t = 3$.

Solution: When $t = 3$, $x = 18$ and $y = 2$.

$\frac{dy}{dx}$ can be found in two (or more!) ways:

$$\begin{array}{l} \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6t^{-2}}{6} \\ \Rightarrow \frac{dy}{dx} = \frac{-1}{t^2} = \frac{-1}{9}, \quad \text{when } t = 3 \end{array} \quad \left| \quad \begin{array}{l} xy = 36 \quad \Rightarrow \quad y = \frac{36}{x} \\ \Rightarrow \frac{dy}{dx} = \frac{-36}{x^2} \\ \Rightarrow \frac{dy}{dx} = \frac{-36}{18^2} = \frac{-1}{9}, \quad \text{when } x = 18 \end{array} \right.$$

$$\Rightarrow \text{equation of tangent is } y - 2 = \frac{-1}{9} (x - 18)$$

$$\Rightarrow x + 9y - 36 = 0$$

Finding a locus

First find expressions for x and y coordinates in terms of a parameter, t or θ , then eliminate the parameter to give an expression involving **only** x and y , which will be the equation of the locus.

Example: The tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, at the point P , $(3 \cos \theta, 4 \sin \theta)$, crosses the x -axis at A , and the y -axis at B .

Find an equation for the locus of the mid-point of AB as P moves round the ellipse, or as θ varies.

Solution: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4 \cos \theta}{-3 \sin \theta}$

$$\Rightarrow \text{equation of tangent is } y - 4 \sin \theta = \frac{4 \cos \theta}{-3 \sin \theta} (x - 3 \cos \theta)$$

$$\Rightarrow 3y \sin \theta + 4x \cos \theta = 12 \cos^2 \theta + 12 \sin^2 \theta = 12$$

Tangent crosses x -axis at A when $y = 0$, $\Rightarrow x = \frac{3}{\cos \theta}$,

and crosses y -axis at B when $x = 0$, $\Rightarrow y = \frac{4}{\sin \theta}$

$$\Rightarrow \text{mid-point of } AB \text{ is } \left(\frac{3}{2 \cos \theta}, \frac{4}{2 \sin \theta} \right) \Leftrightarrow (X, Y)$$

Here $X = \frac{3}{2 \cos \theta}$ and $Y = \frac{4}{2 \sin \theta}$

$$\Rightarrow \cos \theta = \frac{3}{2X} \quad \text{and} \quad \sin \theta = \frac{2}{Y}$$

$$\Rightarrow \text{equation of the locus is } \frac{9}{4X^2} + \frac{4}{Y^2} = 1 \quad \text{since } \cos^2 \theta + \sin^2 \theta = 1$$

$$\text{or } \frac{9}{4x^2} + \frac{4}{y^2} = 1$$

3 Differentiation

Derivatives of hyperbolic functions

$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

and, similarly, $\frac{d(\cosh x)}{dx} = \sinh x$

Also, $y = \tanh x = \frac{\sinh x}{\cosh x}$

$$\Rightarrow \frac{dy}{dx} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

In a similar way, all the derivatives of hyperbolic functions can be found.

$f(x)$	$f'(x)$	
$\sinh x$	$\cosh x$	all positive
$\cosh x$	$\sinh x$	
$\tanh x$	$\operatorname{sech}^2 x$	
$\coth x$	$-\operatorname{cosech}^2 x$	all negative
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$	
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$	

Notice: these are similar to the results for $\sin x$, $\cos x$, $\tan x$ etc., **but** the **minus** signs do not always agree.

The minus signs are ‘*wrong*’ only for $\cosh x$ and $\operatorname{sech} x$ ($= \frac{1}{\cosh x}$).

Derivatives of inverse functions

$$y = \operatorname{arsinh} x$$

$$\Rightarrow \sinh y = x \quad \Rightarrow \quad \cosh y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

$$\Rightarrow \frac{d(\operatorname{arsinh} x)}{dx} = \frac{1}{\sqrt{1+x^2}}$$

The derivatives for other inverse hyperbolic functions can be found in a similar way.

You can also use integration by substitution to find the integrals of the $f'(x)$ column

$f(x)$	$f'(x)$	substitution needed for integration
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$1 - \sin^2 u = \cos^2 u \Rightarrow$ use $x = \sin u$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$1 - \cos^2 u = \sin^2 u \Rightarrow$ use $x = \cos u$
$\arctan x$	$\frac{1}{1+x^2}$	$1 + \tan^2 u = \sec^2 u \Rightarrow$ use $x = \tan u$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$	$1 + \sinh^2 u = \cosh^2 u \Rightarrow$ use $x = \sinh u$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$	$\cosh^2 u - 1 = \sinh^2 u \Rightarrow$ use $x = \cosh u$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$	$1 - \tanh^2 u = \operatorname{sech}^2 u \Rightarrow$ use $x = \tanh u$
$\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$	$\frac{1}{1-x^2}$	partial fractions, see example below

Note that $\int \frac{1}{1-x^2} dx = \frac{1}{2} \int \frac{1}{1+x} + \frac{1}{1-x} dx = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) + c$

With chain rule, product rule and quotient rule you should be able to handle a large variety of combinations of functions.

4 Integration

Standard techniques

Recognise a standard function

Examples: $\int \sec x \tan x \, dx = \sec x + c$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$$

Using formulae to change the integrand

Examples: $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + c$

$$\int \cos^2 x \, dx = \frac{1}{2} \int 1 + \cos 2x \, dx = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) + c$$

$$\int \sinh^2 x \, dx = \frac{1}{2} \int \cosh 2x - 1 \, dx = \frac{1}{2} \left(\frac{1}{2} \sinh 2x - x \right) + c$$

Reverse chain rule

Notice the chain rule pattern, guess an answer and differentiate to find the constant.

Example: $\int \cos^2 x \sin x \, dx$ 'looks like' $u^2 \frac{du}{dx}$ so try $u^3 \Leftrightarrow \cos^3 x$

$$\frac{d(\cos^3 x)}{dx} = 3\cos^2 x (-\sin x) = -3\cos^2 x \sin x \quad \text{so divide by } -3$$

$$\Rightarrow \int \cos^2 x \sin x \, dx = -\frac{1}{3} \cos^3 x + c$$

Example: $\int x^2 (2x^3 - 7)^4 \, dx$ 'looks like' $u^4 \frac{du}{dx}$ so try $u^5 \Leftrightarrow (2x^3 - 7)^5$

$$\frac{d(2x^3 - 7)^5}{dx} = 5(2x^3 - 7)^4 \times 6x^2 = 30x^2(2x^3 - 7)^4 \quad \text{so divide by } 30$$

$$\Rightarrow \int x^2 (2x^3 - 7)^4 \, dx = \frac{1}{30} (2x^3 - 7)^5 + c$$

Example: $\int \operatorname{sech}^4 x \tanh x \, dx$

$$= \int \operatorname{sech}^3 x (\operatorname{sech} x \tanh x) \, dx \quad \text{'looks like' } u^3 \frac{du}{dx} \text{ so try } u^4 = \operatorname{sech}^4 x$$

$$\frac{d(\operatorname{sech}^4 x)}{dx} = -4 \operatorname{sech}^3 x \operatorname{sech} x \tanh x \quad \text{so divide by } -4$$

$$\Rightarrow \int \operatorname{sech}^4 x \tanh x \, dx = -\frac{1}{4} \operatorname{sech}^4 x + c$$

Standard substitutions

$$\int \frac{1}{a^2 + b^2 x^2} dx \quad bx = a \tan u \quad \text{better than } bx = a \sinh u \text{ when there is no } \sqrt{\quad}$$

$$\int \frac{1}{\sqrt{a^2 + b^2 x^2}} dx \quad bx = a \sinh u \quad \text{better than } bx = a \tan u \text{ when there is } \sqrt{\quad}$$

$$\int \frac{1}{a^2 - b^2 x^2} dx \quad bx = a \tanh u \quad \text{or use partial fractions}$$

$$\int \frac{1}{\sqrt{b^2 x^2 - a^2}} dx \quad bx = a \cosh u \quad \text{better than } bx = a \sec u \text{ when there is } \sqrt{\quad}$$

For more complicated integrals like

$$\int \frac{1}{px^2 + qx + r} dx \quad \text{or} \quad \int \frac{1}{\sqrt{px^2 + qx + r}} dx$$

complete the square to give $p(x + a)^2 + b$ and then use a substitution similar to one of the four above.

$$\begin{aligned} \text{Example: } \int \frac{1}{\sqrt{4x^2 - 8x - 5}} dx & \quad 4x^2 - 8x - 5 = 4(x^2 - 2x + 1) - 9 = 4(x - 1)^2 - 9 \\ & = \int \frac{1}{\sqrt{4(x-1)^2 - 9}} dx \end{aligned}$$

$$\text{Substitute } 2(x - 1) = 3 \cosh u \Rightarrow 2 dx = 3 \sinh u du$$

$$= \int \frac{1}{\sqrt{9(\cosh^2 u - 1)}} \frac{3 \sinh u}{2} du$$

$$= \frac{1}{2} \int du = u + c = \frac{1}{2} \operatorname{arcosh} \left(\frac{2x-2}{3} \right) + c$$

Nice trick

$$\text{Example: } I = \int \frac{1}{\sqrt{(2x-3)^2 + 25}} dx$$

$$\text{Solution: } \text{Substitute } u = 2x - 3, \Rightarrow dx = \frac{1}{2} du$$

$$\Rightarrow I = \int \frac{1}{\sqrt{u^2 + 5^2}} \frac{1}{2} du = \frac{1}{2} \operatorname{arsinh} \left(\frac{u}{5} \right) + c, \quad \text{using standard formula}$$

$$= \frac{1}{2} \operatorname{arsinh} \left(\frac{2x-3}{5} \right) + c$$

Important tip

$$\int \frac{x^n}{\sqrt{a^2 \pm x^2}} dx, \text{ etc., is best done with the substitution}$$

$$(i) \quad u \text{ (or } u^2) = a^2 \pm x^2, \text{ when } n \text{ is } \mathbf{odd},$$

$$\text{or (ii) } \quad \text{a trigonometric or hyperbolic function when } n \text{ is } \mathbf{even}.$$

Integration inverse functions and $\ln x$

To integrate inverse trigonometric or hyperbolic functions and $\ln x$ we use integration by parts with $\frac{dv}{dx} = 0$

Example: Find $\int \arctan x \, dx$

Solution: $I = \int \arctan x \, dx$ take $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$

and $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\Rightarrow I = x \arctan x - \int x \times \frac{1}{1+x^2} \, dx$$

$$\Rightarrow I = \int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + c$$

Example: Find $\int \operatorname{arcosh} x \, dx$

Solution: $I = \int \operatorname{arcosh} x \, dx$ take $u = \operatorname{arcosh} x \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{x^2-1}}$

and $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\Rightarrow I = x \operatorname{arcosh} x - \int x \times \frac{1}{\sqrt{x^2-1}} \, dx$$

$$\Rightarrow I = \int \operatorname{arcosh} x \, dx = x \operatorname{arcosh} x - \sqrt{x^2-1} + c$$

Reduction formulae

The first step in finding a reduction formula is often (but not always) integration by parts (sometimes twice). The following examples show a variety of techniques.

Example 1: $I_n = \int x^n e^x dx$.

- (a) Find a reduction formula,
- (b) Find I_0 ,
- (c) Find I_4

Solution:

- (a) Integrating by parts

$$u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$$

and $\frac{dv}{dx} = e^x \Rightarrow v = e^x$

$$\Rightarrow I_n = x^n e^x - \int nx^{n-1} e^x dx$$

$$\Rightarrow I_n = x^n e^x - nI_{n-1}$$

- (b) $I_0 = \int e^x dx = e^x + c$

- (c) Using the reduction formula

$$\begin{aligned} I_4 &= x^4 e^x - 4I_3 = x^4 e^x - 4(x^3 e^x - 3I_2) \\ &= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2I_1) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(xe^x - I_0) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c \end{aligned}$$

since $I_0 = e^x + c$

Example 2: Find a reduction formula for $I_n = \int \tan^n x dx$.

Solution: $I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx$

$$\Rightarrow I_n = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$\Rightarrow I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}.$$

Example 3: (i) Find a reduction formula for $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

(ii) Use the formula to find $I_6 = \int_0^{\pi/2} \sin^6 x \, dx$

Solution: (i) By splitting $\sin^n x = \sin^{n-1} x \sin x$, we can differentiate $\sin^{n-1} x$ reducing the power, and we can integrate $\sin x$

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$$

$$\text{take } u = \sin^{n-1} x \Rightarrow \frac{du}{dx} = (n-1) \sin^{n-2} x \cos x$$

$$\text{and } \frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\Rightarrow I_n = [-\cos x \sin^{n-1} x]_0^{\pi/2} - \int_0^{\pi/2} -\cos x (n-1) \sin^{n-2} x \cos x \, dx$$

$$= 0 + (n-1) \int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx$$

$$= (n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{n-2} x \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$$

$$\Rightarrow I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}.$$

$$(ii) \quad I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3}{4} I_2 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$$

$$\Rightarrow I_6 = \frac{5}{16} \int_0^{\pi/2} 1 \, dx = \frac{5\pi}{32}.$$

Example 4: Find a reduction formula for $I_n = \int \sec^n x \, dx$.

Solution: By splitting $\sec^n x = \sec^{n-2} x \sec^2 x$, we can differentiate $\sec^{n-2} x$ reducing the power, and we can integrate $\sec^2 x$

$$I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$$

$$\text{take } u = \sec^{n-2} x \Rightarrow \frac{du}{dx} = (n-2) \sec^{n-3} x \sec x \tan x$$

$$\text{and } \frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$\Rightarrow I_n = \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

Example 5: Find a reduction formula for $I_n = \int_{-1}^0 x^n (1+x)^2 dx$.

Solution: We can differentiate x^n reducing the power, and we can integrate $(1+x)^2$

$$I_n = \int_{-1}^0 x^n (1+x)^2 dx \quad \text{take } u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$$

$$\text{and } \frac{dv}{dx} = (1+x)^2 \Rightarrow v = \frac{1}{3}(1+x)^3$$

$$\Rightarrow I_n = \left[x^n \times \frac{1}{3}(1+x)^3 \right]_{-1}^0 - \int_{-1}^0 nx^{n-1} \times \frac{1}{3}(1+x)^3 dx.$$

Writing $(1+x)^3 = (1+x)^2(1+x)$ allows us to write the integral in terms of I_{n-1} and I_n .

Many reduction formulae need a fiddle like this – e.g. $(1-x^2)^{\frac{3}{2}} = (1-x^2)^{\frac{1}{2}}(1-x^2)$

$$\Rightarrow I_n = 0 - \frac{n}{3} \int_{-1}^0 x^{n-1} (1+x)^2 (1+x) dx$$

$$\Rightarrow I_n = -\frac{n}{3} \int_{-1}^0 x^{n-1} (1+x)^2 dx - \frac{n}{3} \int_{-1}^0 x^n (1+x)^2 dx$$

$$\Rightarrow I_n = -\frac{n}{3} I_{n-1} - \frac{n}{3} I_n$$

$$\Rightarrow \frac{n+3}{3} I_n = -\frac{n}{3} I_{n-1}$$

$$\Rightarrow I_n = -\frac{n}{n+3} I_{n-1}$$

Example 6: Find a reduction formula for $I_n = \int_0^{\pi/2} x^n \cos x dx$

Solution: We can differentiate x^n reducing the power, and we can integrate $\cos x$

$$I_n = \int_0^{\pi/2} x^n \cos x dx \quad \text{take } u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$$

$$\text{and } \frac{dv}{dx} = \cos x \Rightarrow v = \sin x$$

$$\Rightarrow I_n = [x^n \sin x]_0^{\pi/2} - n \int_0^{\pi/2} \sin x \times x^{n-1} dx$$

Integrating by parts again will change the $\sin x$ to $\cos x$, and reduce the power further.

$$\text{take } u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$$

$$\text{and } \frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \left\{ [x^{n-1}(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} -\cos x \times (n-1)x^{n-2} dx \right\}$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \left\{ 0 + (n-1) \int_0^{\pi/2} x^{n-2} \cos x dx \right\}$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

Example 7: Find a reduction formula for $I_n = \int \frac{\sin nx}{\sin x} dx$

$$\begin{aligned}
 \text{Solution: } I_n &= \int \frac{\sin[(n-2)x+2x]}{\sin x} dx \\
 &= \int \frac{\sin(n-2)x \cos 2x + \cos(n-2)x \sin 2x}{\sin x} dx \\
 &= \int \frac{\sin(n-2)x(1-2\sin^2 x) + \cos(n-2)x \times 2 \sin x \cos x}{\sin x} dx \\
 &= \int \frac{\sin(n-2)x}{\sin x} dx + 2 \int \cos(n-2)x \cos x - \sin(n-2)x \sin x dx \\
 &= I_{n-2} + 2 \int \cos(n-1)x dx \quad \text{using } \cos(A+B) = \cos A \cos B - \sin A \sin B \\
 \Rightarrow I_n &= I_{n-2} + \frac{2}{n-1} \sin(n-1)x.
 \end{aligned}$$

Arc length

All the formulae you need can be remembered from this diagram

arc $PQ \approx$ line segment PQ

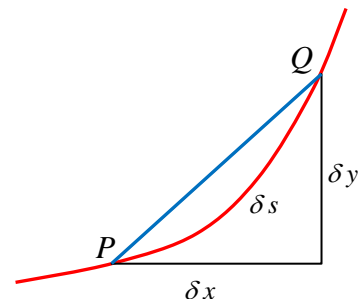
$$\Rightarrow (\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\Rightarrow \left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

and as $\delta x \rightarrow 0$

$$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \text{arc length} = s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



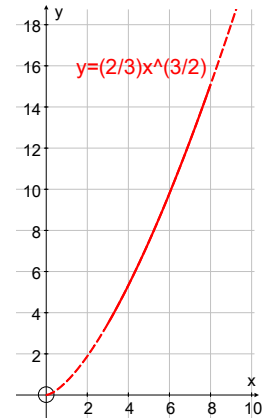
Similarly $\left(\frac{\delta s}{\delta y}\right)^2 \approx \left(\frac{\delta x}{\delta y}\right)^2 + 1 \Rightarrow s = \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$

and $\left(\frac{\delta s}{\delta t}\right)^2 \approx \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$

$\Rightarrow s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ or $s = \int \sqrt{\dot{x}^2 + \dot{y}^2} dt$

for parametric equations.

Example 1: Find the length of the curve $y = \frac{2}{3}x^{3/2}$, from the point where $x = 3$ to the point where $x = 8$.



Solution: The equation of the curve is in Cartesian form so we use

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx .$$

$$y = \frac{2}{3}x^{3/2} \Rightarrow \frac{dy}{dx} = \sqrt{x}$$

$$\Rightarrow s = \int_3^8 \sqrt{1+x} dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2} \right]_3^8 = \frac{2}{3} \times (9)^{3/2} - \frac{2}{3} \times (4)^{3/2}$$

$$\Rightarrow s = 12\frac{2}{3} .$$

Example 2: Find the length of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution: The curve is given in parametric form so

we use $s = \int \sqrt{\dot{x}^2 + \dot{y}^2} dt$.

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

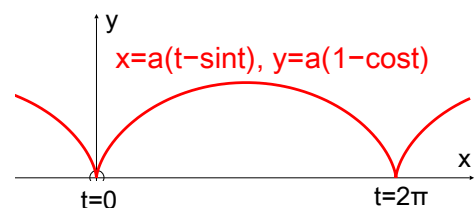
$$\Rightarrow \frac{dx}{dt} = a(1 - \cos t), \text{ and } \frac{dy}{dt} = a \sin t$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = a^2(1 - 2 \cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t)$$

$$\Rightarrow \sqrt{\dot{x}^2 + \dot{y}^2} = a \sqrt{2 \left(1 - \left[1 - 2 \sin^2 \left(\frac{t}{2} \right) \right] \right)} = 2a \sin \left(\frac{t}{2} \right)$$

$$\Rightarrow s = \int_0^{2\pi} 2a \sin \left(\frac{t}{2} \right) dt$$

$$\Rightarrow s = \left[-4a \cos \left(\frac{t}{2} \right) \right]_0^{2\pi} = 4a - - 4a = 8a.$$

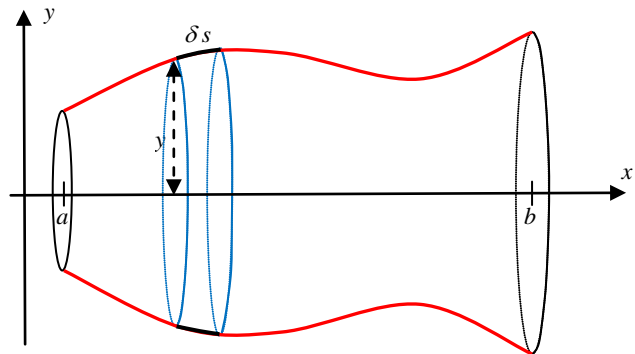


Area of a surface of revolution

A curve is rotated about the x -axis.

To find the area of the surface formed between $x = a$ and $x = b$, we consider a small section of the curve, δs , at a distance of y from the x -axis.

When this small section is rotated about the x -axis, the shape formed is approximately a cylinder of radius y and length δs .



The surface area of this (cylindrical) shape $\approx 2\pi y \delta s$

\Rightarrow The total surface area $\approx \sum_a^b 2\pi y \delta s$

and, as $\delta s \rightarrow 0$, the area of the surface is $A = \int_a^b 2\pi y ds$.

And so $A = \int_a^b 2\pi y \frac{ds}{dx} dx$ or $A = \int_a^b 2\pi y \frac{ds}{dt} dt$

We can use $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ or $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, as appropriate,

remembering that $(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$

Example 1: A sphere has radius r . Find the surface area of the sphere between the planes $x = a$ and $x = b$.

Solution: The Cartesian form is most suitable here.

$$A = \int_a^b 2\pi y \frac{ds}{dx} dx$$

$$x^2 + y^2 = r^2$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

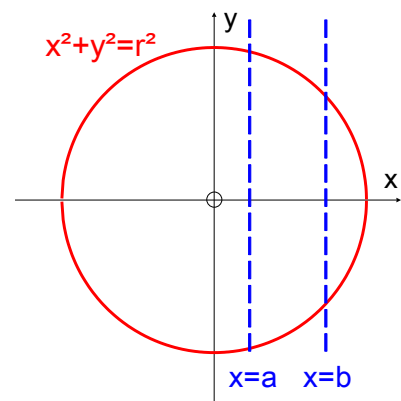
$$\text{and } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow A = \int_a^b 2\pi y \sqrt{1 + \frac{x^2}{y^2}} dx = \int_a^b 2\pi \sqrt{y^2 + x^2} dx$$

$$= \int_a^b 2\pi r dx \quad \text{since } x^2 + y^2 = r^2$$

$$\Rightarrow A = [2\pi r x]_a^b = 2\pi r(b - a) \quad \text{since } r \text{ is constant}$$

Notice that the area of the whole sphere is from $a = -r$ to $b = r$ giving surface area of a sphere is $4\pi r^2$.

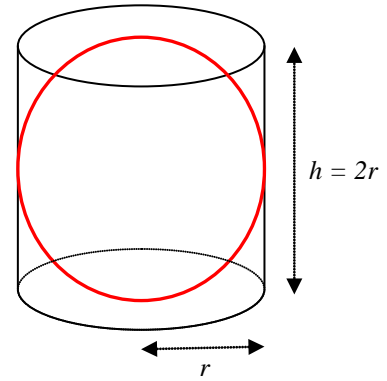


Historical note.

Archimedes showed that the area of a sphere is equal to the area of the curved surface of the surrounding cylinder.

Thus the area of the sphere is

$$A = 2\pi rh = 4\pi r^2 \quad \text{since } h = 2r.$$



Example 2: The parabola, $x = at^2$, $y = 2at$, between the origin ($t = 0$) and P ($t = 2$) is rotated about the x -axis.

Find the surface area of the shape formed.

Solution: The parametric form is suitable here.

$$A = \int_a^b 2\pi y \frac{ds}{dt} dt$$

and
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

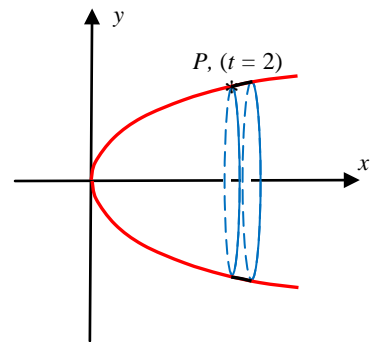
$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{(2at)^2 + (2a)^2} = 2a\sqrt{t^2 + 1}$$

$$\Rightarrow A = \int_0^2 2\pi 2at \times 2a\sqrt{t^2 + 1} dt$$

$$= 8\pi a^2 \times \frac{1}{3} \left[(t^2 + 1)^{3/2} \right]_0^2$$

$$\Rightarrow A = \frac{8\pi a^2}{3} (5^{3/2} - 1)$$



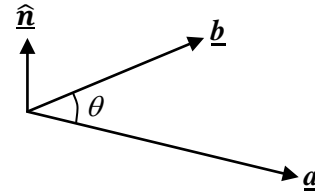
5 Vectors

Vector product

The vector, or cross, product of \underline{a} and \underline{b} is

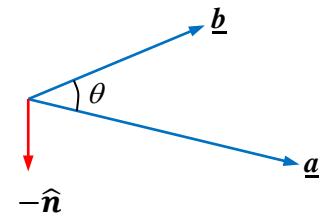
$$\underline{a} \times \underline{b} = ab \sin\theta \hat{n}$$

where \hat{n} is a *unit* (length 1) vector which is *perpendicular* to both \underline{a} and \underline{b} , and θ is the angle between \underline{a} and \underline{b} .



The direction of \hat{n} is that in which a right hand corkscrew would move when turned through the angle θ from \underline{a} to \underline{b} .

Notice that $\underline{b} \times \underline{a} = ab \sin\theta(-\hat{n})$, where $-\hat{n}$ is in the opposite direction to \hat{n} , since the corkscrew would move in the opposite direction when moving from \underline{b} to \underline{a} .



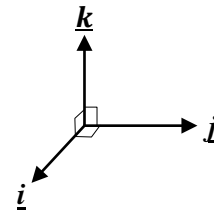
Thus $\underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$.

The vectors $\underline{i}, \underline{j}$ and \underline{k}

For unit vectors, $\underline{i}, \underline{j}$ and \underline{k} , in the directions of the axes

$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j},$$

$$\underline{i} \times \underline{k} = -\underline{j}, \quad \underline{j} \times \underline{i} = -\underline{k}, \quad \underline{k} \times \underline{j} = -\underline{i}.$$



Properties

$$\underline{a} \times \underline{a} = \underline{0}$$

since $\theta = 0$

$$\underline{a} \times \underline{b} = \underline{0} \Rightarrow \underline{a} \text{ is parallel to } \underline{b}$$

since $\sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$

$$\text{or } \underline{a} \text{ or } \underline{b} = \underline{0}$$

$$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$$

remember the brilliant demo with the straws!

$$\underline{a} \times \underline{b} \text{ is perpendicular to both } \underline{a} \text{ and } \underline{b}$$

from the definition

Component form

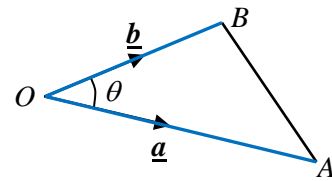
Using the above we can show that

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Applications of the vector product

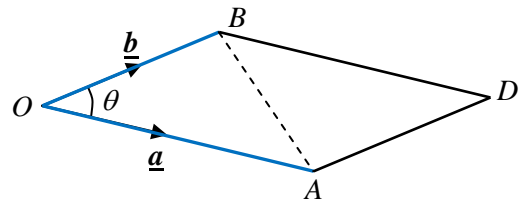
$$\text{Area of triangle } OAB = \frac{1}{2} ab \sin \theta$$

$$\Rightarrow \text{area of triangle } OAB = \frac{1}{2} |\underline{a} \times \underline{b}|$$



Area of parallelogram $OADB$ is twice the area of the triangle OAB

$$\Rightarrow \text{area of parallelogram } OADB = |\underline{a} \times \underline{b}|$$



Example: A is $(-1, 2, 1)$, B is $(2, 3, 0)$ and C is $(3, 4, -2)$.

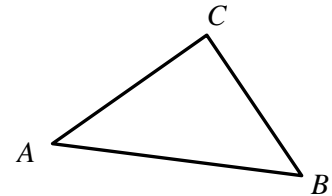
Find the area of the triangle ABC .

Solution: The area of the triangle $ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & -1 \\ 4 & 2 & -3 \end{vmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{area } ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{1^2 + 5^2 + 3^2} = \frac{1}{2} \sqrt{35}$$



Volume of a parallelepiped

In the parallelepiped,

the base is parallel to \underline{b} and \underline{c}

$\hat{\underline{n}}$ is a unit vector perpendicular to the base

and the height $\underline{h} = h \hat{\underline{n}}$,

where $h = \pm a \cos \phi = \pm \underline{a} \cdot \hat{\underline{n}}$

\pm because ϕ might be obtuse

The area of base = $bc \sin \theta$

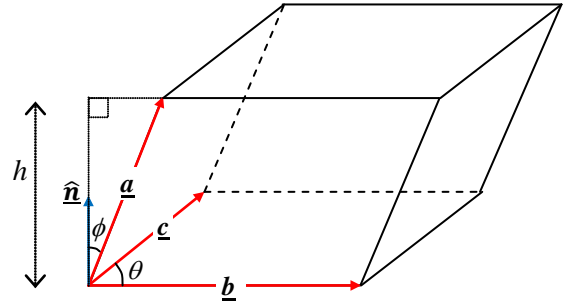
\Rightarrow volume $V = \pm h \times bc \sin \theta$

$\Rightarrow \pm V = a \cos \phi \times bc \sin \theta$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{a} \cdot (bc \sin \theta \hat{\underline{n}}) = \underline{a} \cdot \hat{\underline{n}} (bc \sin \theta)$$

$\Rightarrow \underline{a} \cdot (\underline{b} \times \underline{c}) = a \cos \phi \times bc \sin \theta = \pm V$

\Rightarrow volume of parallelepiped = $|\underline{a} \cdot (\underline{b} \times \underline{c})|$



Triple scalar product

$$\begin{aligned} |\underline{a} \cdot (\underline{b} \times \underline{c})| &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_2 c_3 - b_3 c_2 \\ -b_1 c_3 + b_3 c_1 \\ b_1 c_2 - b_2 c_1 \end{pmatrix} \\ &= a_1(b_2 c_3 - b_3 c_2) + a_2(-b_1 c_3 + b_3 c_1) + a_3(b_1 c_2 - b_2 c_1) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

By expanding the determinants we can show that

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} \quad \text{keep the order of } \underline{a}, \underline{b}, \underline{c} \text{ but change the order of the } \times \text{ and } \cdot$$

For this reason the triple scalar product is written as $\{\underline{a}, \underline{b}, \underline{c}\}$

$$\{\underline{a}, \underline{b}, \underline{c}\} = \underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$$

It can also be shown that a cyclic change of the order of $\underline{a}, \underline{b}, \underline{c}$ does not change the value, but interchanging two of the vectors multiplies the value by -1 .

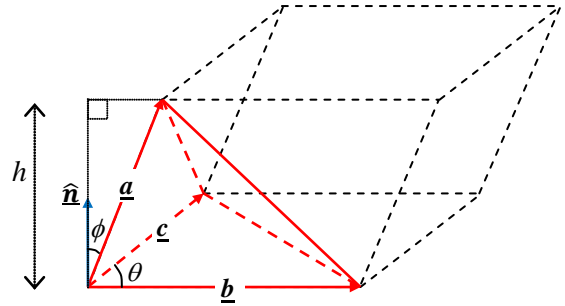
$$\Rightarrow \{\underline{a}, \underline{b}, \underline{c}\} = \{\underline{c}, \underline{a}, \underline{b}\} = \{\underline{b}, \underline{c}, \underline{a}\} = -\{\underline{a}, \underline{c}, \underline{b}\} = -\{\underline{c}, \underline{b}, \underline{a}\} = -\{\underline{b}, \underline{a}, \underline{c}\}$$

Volume of a tetrahedron

The volume of a tetrahedron is

$$\frac{1}{3} \text{ Area of base} \times h$$

The height of the tetrahedron is the same as the height of the parallelepiped, but its base has half the area



$$\Rightarrow \text{volume of tetrahedron} = \frac{1}{6} \text{ volume of parallelepiped}$$

$$\Rightarrow \text{volume of tetrahedron} = \frac{1}{6} |\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}|$$

Example: Find the volume of the tetrahedron $ABCD$,

given that A is $(1, 0, 2)$, B is $(-1, 2, 2)$, C is $(1, 1, -3)$ and D is $(4, 0, 3)$.

Solution: Volume = $\frac{1}{6} |\{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\}|$

$$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\} = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & -5 \\ -2 & 2 & 0 \end{vmatrix} = 3 \times 10 + 2 = 32$$

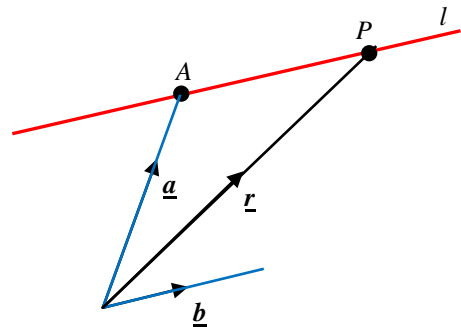
$$\Rightarrow \text{volume of tetrahedron is } \frac{1}{6} \times 32 = 5\frac{1}{3}$$

Equations of straight lines

Vector equation of a line

$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ is the equation of a line through the point A and parallel to the vector \mathbf{b} ,

$$\text{or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$



Cartesian equation of a line in 3-D

Eliminating λ from the above equation we obtain

$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} = \frac{z-n}{\gamma} \quad (= \lambda)$$

is the equation of a line through the point (l, m, n) and parallel to the vector $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

This strange form of equation is really the intersection of the planes

$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} \quad \text{and} \quad \frac{y-m}{\beta} = \frac{z-n}{\gamma} \quad \left(\text{and} \quad \frac{x-l}{\alpha} = \frac{z-n}{\gamma} \right).$$

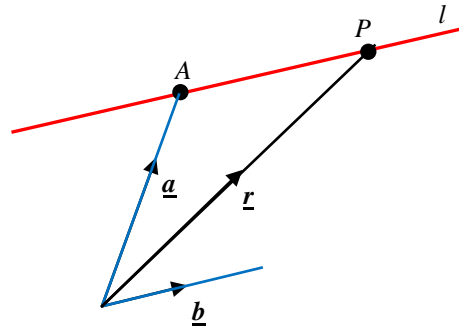
Vector product equation of a line

$\overrightarrow{AP} = \underline{r} - \underline{a}$ and is parallel to the vector \underline{b}

$$\Rightarrow \overrightarrow{AP} \times \underline{b} = \underline{0}$$

$\Rightarrow (\underline{r} - \underline{a}) \times \underline{b} = \underline{0}$ is the equation of a line through A and parallel to \underline{b} .

or $\underline{r} \times \underline{b} = \underline{a} \times \underline{b} = \underline{c}$ is the equation of a line parallel to \underline{b} .



Notice that all three forms of equation refer to a line through the point A and parallel to the vector \underline{b} .

Example: A straight line has Cartesian equation

$$x = \frac{2y+4}{5} = \frac{3-z}{2}.$$

Find its equation (i) in the form $\underline{r} = \underline{a} + \lambda \underline{b}$, (ii) in the form $\underline{r} \times \underline{b} = \underline{c}$.

Solution:

First re-write the equation in the *standard* manner

$$\Rightarrow \frac{x-0}{1} = \frac{y-(-2)}{2.5} = \frac{z-3}{-2}$$

\Rightarrow the line passes through $A, (0, -2, 3)$, and is parallel to $\underline{b}, \begin{pmatrix} 1 \\ 2.5 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$

$$(i) \quad \underline{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$$

$$(ii) \quad \left(\underline{r} - \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right) \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 1 \\ 2.5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -2 & 3 \\ 2 & 5 & -4 \end{vmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}.$$

Equation of a plane

Scalar product form

Let \underline{n} be a vector perpendicular to the plane π .

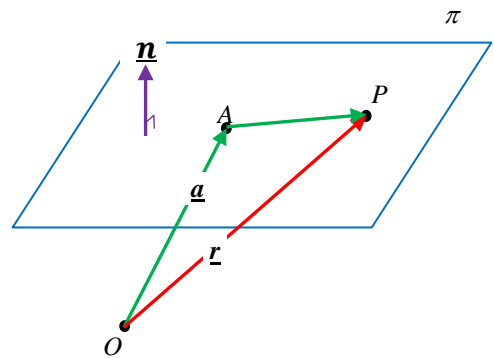
Let A be a fixed point in the plane, and P be a general point, (x, y, z) , in the plane.

Then \overrightarrow{AP} is parallel to the plane, and therefore perpendicular to \underline{n}

$$\Rightarrow \quad \overrightarrow{AP} \cdot \underline{n} = 0 \quad \Rightarrow \quad (\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

$$\Rightarrow \quad \underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} = \text{a constant, } d$$

$$\Rightarrow \quad \underline{r} \cdot \underline{n} = d \text{ is the equation of a plane perpendicular to the vector } \underline{n}.$$



Cartesian form

$$\text{If } \underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } \underline{r} \cdot \underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz$$

$$\Rightarrow \quad ax + by + cz = d \text{ is the Cartesian equation of a plane perpendicular to } \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Example: Find the scalar product form and the Cartesian equation of the plane through the points $A, (3, 2, 5), B, (-1, 0, 3)$ and $C, (2, 1, -2)$.

Solution: We first need a vector perpendicular to the plane.

$A, (3, 2, 5), B, (-1, 0, 3)$ and $C, (2, 1, -2)$ lie in the plane

$$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix} \text{ are parallel to the plane}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} \text{ is perpendicular to the plane}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & -2 & -2 \\ -1 & -1 & -7 \end{vmatrix} = \begin{pmatrix} 12 \\ -26 \\ 2 \end{pmatrix} = 2 \times \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix} \quad \text{using smaller numbers}$$

$$\Rightarrow 6x - 13y + z = d$$

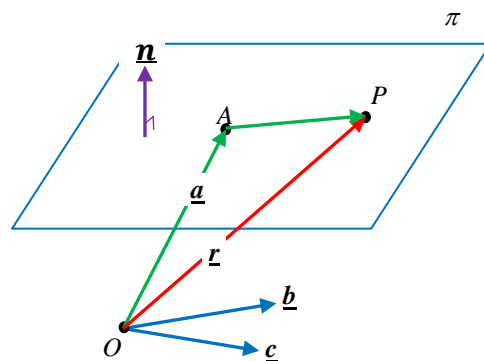
$$\text{but } A, (3, 2, 5) \text{ lies in the plane} \Rightarrow d = 6 \times 3 - 13 \times 2 + 5 = -3$$

$$\Rightarrow \text{Cartesian equation is } 6x - 13y + z = -3$$

$$\text{and scalar product equation is } \underline{r} \cdot \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix} = -3.$$

Vector equation of a plane

$\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$ is the equation of a plane, π , through A and parallel to the vectors \underline{b} and \underline{c} .



Example: Find the vector equation of the plane through the points $A, (1, 4, -2), B, (1, 5, 3)$ and $C, (4, 7, 2)$.

Solution: We want the plane through $A, (1, 4, -2)$, parallel to $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

$$\Rightarrow \text{vector equation is } \underline{r} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}.$$

Distance from a point to a plane

Example: Find the distance from the point $P(-2, 3, 5)$ to the plane $4x - 3y + 12z = 21$.

Solution: Let M be the foot of the perpendicular from P to the plane. The distance of the origin from the plane is PM .

We must first find the intersection of the line PM with the plane.

PM is perpendicular to the plane
and so is parallel to $\underline{n} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$.

$$\Rightarrow \text{the line } PM \text{ is } \underline{r} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{pmatrix} -2 + 4\lambda \\ 3 - 3\lambda \\ 5 + 12\lambda \end{pmatrix},$$

and the point of intersection of PM with the plane is given by

$$4(-2 + 4\lambda) - 3(3 - 3\lambda) + 12(5 + 12\lambda) = 21$$

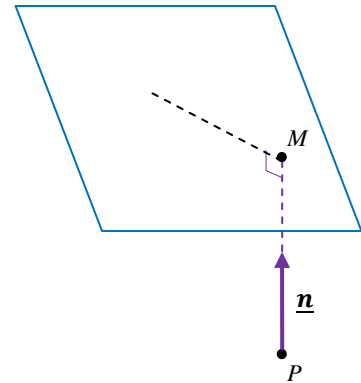
$$\Rightarrow -8 + 16\lambda - 9 + 9\lambda + 60 + 144\lambda = 21$$

$$\Rightarrow \lambda = \frac{-22}{169}$$

$$\Rightarrow \overrightarrow{PM} = \frac{-22}{169} \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$$

$$\Rightarrow \text{distance} = |\overrightarrow{PM}| = \frac{22}{169} \sqrt{4^2 + 3^2 + 12^2} = \frac{22}{13}$$

The distance of the P from the plane is $\frac{22}{13}$.



Distance from any point to a plane

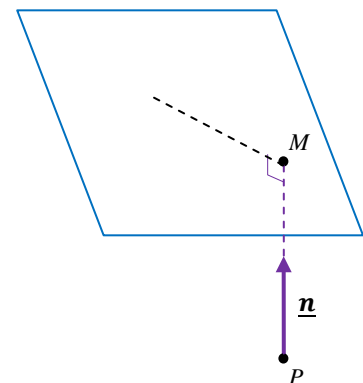
The above technique can be used to find the formula:-

distance, s , from the point $P(\alpha, \beta, \gamma)$ to the plane

$n_1x + n_2y + n_3z + d = 0$ is given by

$$s = \left| \frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \right|$$

This formula is in your formula booklets, but **not** in your text books.



Reflection of a point in a plane

Example: Find the reflection of the point $A(10, 1, 7)$ in the plane π , $\underline{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 7$.

Solution: Find the point of intersection, P , of the line through A and perpendicular to π with the plane π . Then find \overrightarrow{AP} , to give $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP}$.

Line through A perpendicular to π is

$$\underline{r} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

This meets the plane π when

$$3(10+3\lambda) - 2(1-2\lambda) + (7+\lambda) = 7$$

$$\Rightarrow 30 + 9\lambda - 2 + 4\lambda + 7 + \lambda = 7$$

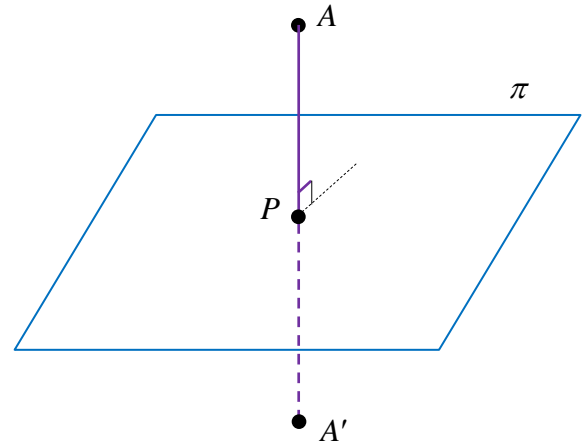
$$\Rightarrow \lambda = -2$$

$$\Rightarrow \overrightarrow{OP} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + (-2) \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-2) \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 3 \end{pmatrix}$$

$$\Rightarrow \text{the reflection of } A \text{ is } A', (-2, 9, 3)$$



Distance between parallel planes

Example: Find the distance between the parallel planes

$$\pi_1: 2x - 6y + 3z = 9 \quad \text{and} \quad \pi_2: 2x - 6y + 3z = 5$$

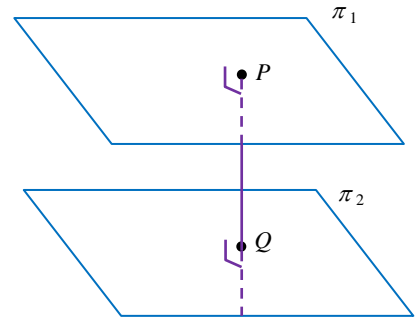
Solution: Take any point, P , on one of the planes, and then use the above formula for the shortest distance, PQ , between the planes.

By inspection the point $P(0, 0, 3)$ lies on π_1

$$\Rightarrow \text{shortest distance } s \text{ from } P \text{ to the plane } \pi_2 \text{ is } \left| \frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \right|$$

$$\Rightarrow \text{shortest distance } s = \left| \frac{2 \times 0 - 6 \times 0 + 3 \times 3 - 5}{\sqrt{2^2 + 6^2 + 3^2}} \right| = \frac{4}{7}$$

The distance between the planes is $\frac{4}{7}$.

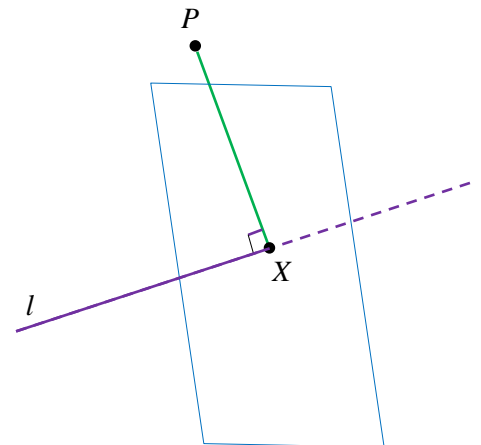


Shortest distance from a point to a line

Example: Find the shortest distance from the point

$$P(3, -2, 4) \text{ to the line } l, \quad \underline{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

Solution: Any plane $2x - 3y + 6z = d$ must be perpendicular to the line l . If we make this plane pass through P and if it meets the line l in the point X , then PX must be perpendicular to the line l , and so PX is the shortest distance from P to the line l .



Plane passes through $P(3, -2, 4)$

$$\Rightarrow 2x - 3y + 6z = 2 \times 3 - 3 \times (-2) + 6 \times 4 = 36$$

$$\Rightarrow 2x - 3y + 6z = 36$$

$$l \text{ meets plane} \Rightarrow 2(-2 + 2\lambda) - 3(3 - 3\lambda) + 6(6\lambda) = 36$$

$$\Rightarrow -4 + 4\lambda - 9 + 9\lambda + 36\lambda = 36 \quad \Rightarrow \lambda = 1$$

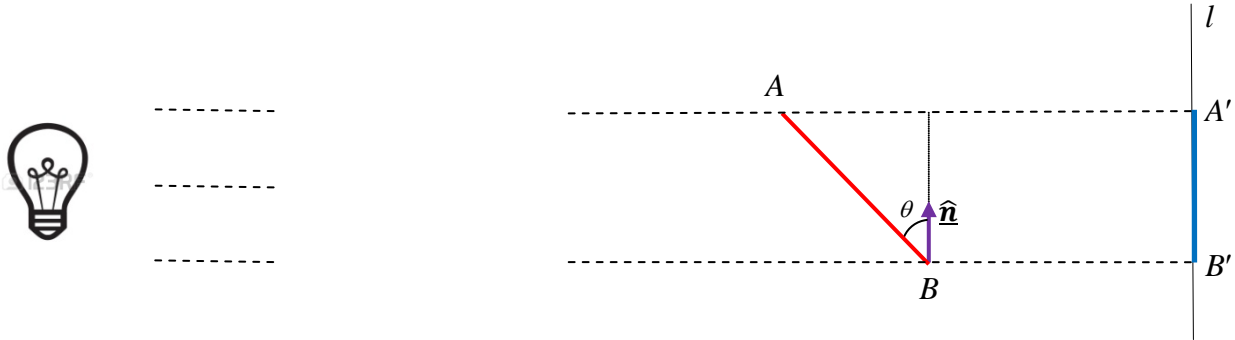
$\Rightarrow X$ is the point $(-2, 0, 6)$

$$\overrightarrow{PX} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{shortest distance is } PX = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$$

Projections – an alternative approach

Imagine a light bulb causing a rod, AB , to make a shadow, $A'B'$, on the line l . If the light bulb is far enough away, we can think of all the light rays as parallel, and, if the rays are all perpendicular to the line l , the shadow is the *projection* of the rod onto l (strictly speaking an *orthogonal projection*).



The length of the shadow, $B'A'$, is $|BA \cos \theta| = |\overrightarrow{BA} \cdot \underline{\hat{n}}|$, where $\underline{\hat{n}}$ is a unit vector parallel to the line l .

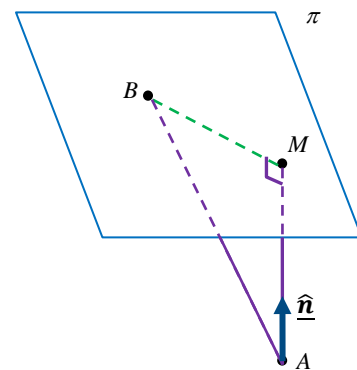
Modulus signs are needed in case $\underline{\hat{n}}$ is in the opposite direction.

Shortest distance from a point from a plane.

To find AM , the shortest distance from A to the plane π ,

For any point, B , on π AM is the projection of AB onto the line AM

$$\Rightarrow AM = |\overrightarrow{AB} \cdot \underline{\hat{n}}|$$



Example: Find the shortest distance from the point $A (-2, 3, 5)$ to the plane $4x - 3y + 12z = 21$.

Solution: By inspection $B (0, -7, 0)$ lies on the plane

$$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} 0 \\ -7 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix}$$

$$\underline{n} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \Rightarrow n = \sqrt{4^2 + 3^2 + 12^2} = 13$$

$$\Rightarrow \text{shortest distance} = |\overrightarrow{AB} \cdot \underline{\hat{n}}| = \left| \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix} \cdot \frac{1}{13} \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{22}{13}$$

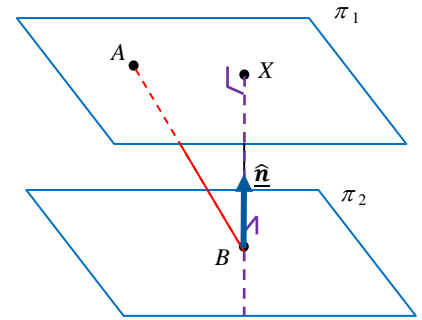
Distance between parallel planes

Example: Find the distance between the parallel planes

$$\pi_1: 2x - 6y + 3z = 9 \quad \text{and} \quad \pi_2: 2x - 6y + 3z = 5$$

Solution: Take any point, B , on one of the planes, π_2 , and then consider the line BX perpendicular to both planes; BX is then the shortest distance between the planes.

Then choose any point, A , on π_1 , and BX is now the projection of AB onto BX



$$\Rightarrow \text{shortest distance} = BX = |\overrightarrow{AB} \cdot \hat{\mathbf{n}}|$$

or $\text{shortest distance} = |(\underline{\mathbf{b}} - \underline{\mathbf{a}}) \cdot \hat{\mathbf{n}}|$, for any two points A and B , one on each plane, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both planes.

By inspection the point $A(0, 0, 3)$ lies on π_1 , and the point $B(2.5, 0, 0)$ lies on π_2

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2.5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.5 \\ 0 \\ 3 \end{pmatrix}$$

$$\underline{\mathbf{n}} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \Rightarrow n = \sqrt{2^2 + 6^2 + 3^2} = 7$$

$$\Rightarrow \text{shortest distance} = \left| \begin{pmatrix} -2.5 \\ 0 \\ 3 \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \right| = \frac{4}{7}$$

Shortest distance between two skew lines

It can be shown that there must be a line joining two skew lines which is perpendicular to both lines.

This line is XY and is the shortest distance between the lines.

The vector $\underline{n} = \underline{b} \times \underline{d}$ is perpendicular to both lines

\Rightarrow the unit vector $\hat{\underline{n}} = \frac{\underline{b} \times \underline{d}}{|\underline{b} \times \underline{d}|}$

Now imagine two parallel planes π_1 and π_2 , both perpendicular to $\hat{\underline{n}}$, one containing the line l_1 and the other containing the line l_2 .

A and C are points on l_1 and l_2 , and therefore on π_1 and π_2 .

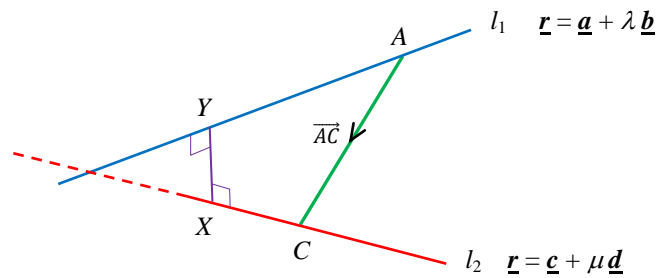
We now have two parallel planes with two points, A and C , one on each plane, and the planes are both perpendicular to $\hat{\underline{n}}$.

As in the example for the distance between parallel planes,

the shortest distance $d = |\overrightarrow{AC} \cdot \hat{\underline{n}}|$

$\Rightarrow d = \left| (\underline{c} - \underline{a}) \cdot \frac{\underline{b} \times \underline{d}}{|\underline{b} \times \underline{d}|} \right|$

This result is not in your formula booklet, SO LEARN IT – please

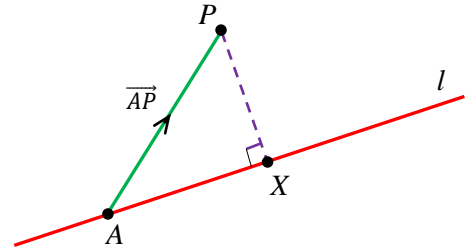


Shortest distance from a point to a line

In trying to find the shortest distance from a point P to a line l , $\underline{r} = \underline{a} + \lambda \underline{b}$, we do not know $\hat{\underline{n}}$, the direction of the line through P perpendicular to l .

Some lateral thinking is needed.

We do know A , a point on the line, and \underline{b} , the direction of the line l



$$\Rightarrow |\overrightarrow{AP} \cdot \hat{\underline{b}}| = AX, \text{ the projection of } AP \text{ onto } l$$

and we can now find $PX = \sqrt{AP^2 - AX^2}$, using Pythagoras

Example: Find the shortest distance from the point $P(3, -2, 4)$

$$\text{to the line } l, \underline{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

Solution: If l is $\underline{r} = \underline{a} + \lambda \underline{b}$, then $\underline{a} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

$$\Rightarrow b = \sqrt{2^2 + 3^2 + 6^2} = 7, \quad \Rightarrow \hat{\underline{b}} = \frac{1}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

$$\text{and } \overrightarrow{AP} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$$

$$\Rightarrow AX = |\overrightarrow{AP} \cdot \hat{\underline{b}}| = \left| \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right| = \frac{10+15+24}{7} = 7$$

$$\begin{aligned} \Rightarrow PX &= \sqrt{AP^2 - AX^2} = \sqrt{(5^2 + 5^2 + 4^2) - 7^2} \\ &= \sqrt{17} \end{aligned}$$

Line of intersection of two planes

Example: Find an equation for the line of intersection of the planes

$$x + y + 2z = 4 \quad \mathbf{I}$$

and $2x - y + 3z = 4 \quad \mathbf{II}$

Solution: Eliminate one variable –

$$\mathbf{I} + \mathbf{II} \Rightarrow 3x + 5z = 8$$

We are *not* expecting a unique solution, so put one variable, z say, equal to λ and find the other variables in terms of λ .

$$z = \lambda \Rightarrow x = \frac{8-5\lambda}{3}$$

$$\mathbf{I} \Rightarrow y = 4 - x - 2z = 4 - \frac{8-5\lambda}{3} - 2\lambda = \frac{4-\lambda}{3}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5/3 \\ -1/3 \\ 1 \end{pmatrix}$$

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$ making the numbers nicer in the **direction vector only**

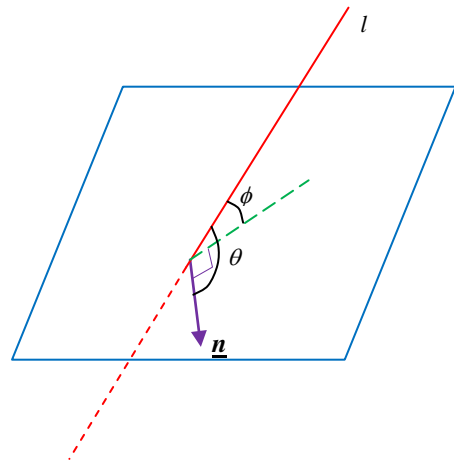
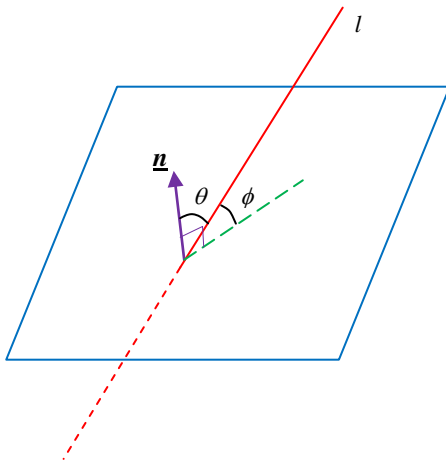
which is the equation of a line through $\left(\frac{8}{3}, \frac{4}{3}, 0\right)$ and parallel to $\begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$.

Angle between line and plane

Let the acute angle between the line and the plane be ϕ .

First find the angle between the line and the normal vector, θ .

There are two possibilities – as shown below:



(i) \underline{n} and the angle ϕ are on the same side of the plane
 $\Rightarrow \phi = 90 - \theta$

(ii) \underline{n} and the angle ϕ are on opposite sides of the plane
 $\Rightarrow \phi = \theta - 90$

Example: Find the angle between the line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $2x + 3y - 7z = 5$.

Solution: The line is parallel to $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, and the normal vector to the plane is $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$.

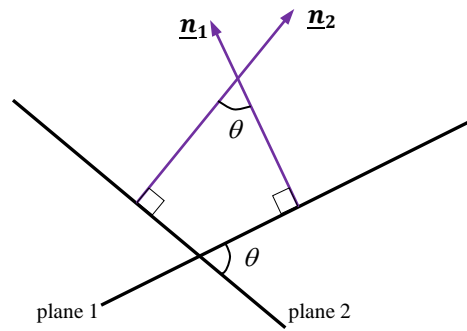
$$\underline{a} \cdot \underline{b} = ab \cos \theta \Rightarrow 21 = \sqrt{2^2 + 1^2 + 2^2} \sqrt{2^2 + 3^2 + 7^2} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{7}{\sqrt{62}} \Rightarrow \theta = 27.3^\circ$$

$$\Rightarrow \text{the angle between the line and the plane, } \phi = 90 - 27.3 = 62.7^\circ$$

Angle between two planes

If we look 'end-on' at the two planes, we can see that the angle between the planes, θ , equals the angle between the normal vectors.



Example: Find the angle between the planes

$$2x + y + 3z = 5 \quad \text{and} \quad 2x + 3y + z = 7$$

Solution: The normal vectors are $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$$\underline{a} \cdot \underline{b} = ab \cos \theta \Rightarrow 10 = \sqrt{2^2 + 1^2 + 3^2} \times \sqrt{2^2 + 1^2 + 3^2} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{10}{14} \Rightarrow \theta = 44.4^\circ$$

6 Matrices

Basic definitions

Dimension of a matrix

A matrix with r rows and c columns has *dimension* $r \times c$.

Transpose and symmetric matrices

The *transpose*, A^T , of a matrix, A , is found by interchanging rows and columns

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$(AB)^T = B^T A^T \quad - \text{ note the change of order of } A \text{ and } B.$$

A matrix, S , is *symmetric* if the elements are symmetrically placed about the leading diagonal,

or if $S = S^T$.

$$\text{Thus, } S = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \text{ is a symmetric matrix.}$$

Identity and zero matrices

$$\text{The } \textit{identity} \text{ matrix } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{and the } \textit{zero} \text{ matrix is } \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Determinant of a 3×3 matrix

The *determinant* of a 3×3 matrix, A , is

$$\det(A) = \Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\Rightarrow \Delta = aei - afh - bdi + bfg + cdh - ceg$$

Properties of the determinant

- 1) A determinant can be expanded by any row or column using $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

e.g. $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$ using the middle row and leaving the value unchanged

- 2) Interchanging two rows changes the sign of the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

which can be shown by evaluating both determinants

- 3) A determinant with two identical rows (or columns) has value 0.

$$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

interchanging the two identical rows gives $\Delta = -\Delta \Rightarrow \Delta = 0$

- 4) $\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$ this can be shown by multiplying out

Singular and non-singular matrices

A matrix, \mathbf{A} , is *singular* if its determinant is zero, $\det(\mathbf{A}) = 0$

A matrix, \mathbf{A} , is *non-singular* if its determinant is not zero, $\det(\mathbf{A}) \neq 0$

Inverse of a 3×3 matrix

This is tedious, but no reason to make a mistake if you are careful.

Cofactors

In $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the cofactors of a, b, c , etc. are A, B, C etc., where

$$A = + \begin{vmatrix} e & f \\ h & i \end{vmatrix}, \quad B = - \begin{vmatrix} d & f \\ g & i \end{vmatrix}, \quad C = + \begin{vmatrix} d & e \\ g & h \end{vmatrix},$$

$$D = - \begin{vmatrix} b & c \\ h & i \end{vmatrix}, \quad E = + \begin{vmatrix} a & c \\ g & i \end{vmatrix}, \quad F = - \begin{vmatrix} a & b \\ g & h \end{vmatrix},$$

$$G = + \begin{vmatrix} b & c \\ e & f \end{vmatrix}, \quad H = - \begin{vmatrix} a & c \\ d & f \end{vmatrix}, \quad I = + \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

These are the 2×2 matrices used in finding the determinant, together with the correct

sign from $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Finding the inverse

- 1) Find the determinant, $\det(A)$.

If $\det(A) = 0$, then A is singular and has no inverse.

- 2) Find the matrix of cofactors $C = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$

- 3) Find the transpose of C , $C^T = \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

- 4) Divide C^T by $\det(A)$ to give $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

See example 10 on page 148.

Properties of the inverse

- 1) $A^{-1}A = AA^{-1} = I$

- 2) $(AB)^{-1} = B^{-1}A^{-1}$

- note the change of order of A and B .

Proof $(AB)^{-1}AB = I$

from definition of inverse

$$\Rightarrow (AB)^{-1}AB(B^{-1}A^{-1}) = I(B^{-1}A^{-1})$$

$$\Rightarrow (AB)^{-1}A(BB^{-1})A^{-1} = B^{-1}A^{-1} \quad \Rightarrow (AB)^{-1}AIA^{-1} = B^{-1}A^{-1}$$

$$\Rightarrow (AB)^{-1}AA^{-1} = B^{-1}A^{-1} \quad \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

- 3) $\det(A^{-1}) = \frac{1}{\det(A)}$

Matrices and linear transformations

Linear transformations

T is a linear transformation on a set of vectors if

- (i) $T(\underline{x}_1 + \underline{x}_2) = T(\underline{x}_1) + T(\underline{x}_2)$ for all vectors \underline{x} and \underline{y}
- (ii) $T(k\underline{x}) = kT(\underline{x})$ for all vectors \underline{x}

Example: Show that $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ -z \end{pmatrix}$ is a linear transformation.

Solution:

- (i)
$$T(\underline{x}_1 + \underline{x}_2) = T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 2(x_1 + x_2) \\ x_1 + x_2 + y_1 + y_2 \\ -z_1 - z_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1 + y_1 \\ -z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2 + y_2 \\ -z_2 \end{pmatrix} = T(\underline{x}_1) + T(\underline{x}_2)$$
$$\Rightarrow T(\underline{x}_1 + \underline{x}_2) = T(\underline{x}_1) + T(\underline{x}_2)$$
- (ii)
$$T(k\underline{x}) = T\left(k \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} = \begin{pmatrix} 2kx \\ kx + ky \\ -kz \end{pmatrix} = k \begin{pmatrix} 2x_1 \\ x_1 + y_1 \\ -z_1 \end{pmatrix} = kT(\underline{x})$$
$$\Rightarrow T(k\underline{x}) = kT(\underline{x})$$

Both (i) and (ii) are satisfied, and so T is a linear transformation.

All matrices can represent linear transformations.

Base vectors \underline{i} , \underline{j} , \underline{k}

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Under the transformation with matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ d \\ g \end{pmatrix} \quad \text{the first column of the matrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} b \\ e \\ h \end{pmatrix} \quad \text{the second column of the matrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} c \\ f \\ i \end{pmatrix} \quad \text{the third column of the matrix}$$

This is an important result, as it allows us to find the matrix for given transformations.

Example: Find the matrix for a reflection in the plane $y = x$

Solution: The z -axis lies in the plane $y = x$ so $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

\Rightarrow the third column of the matrix is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Also $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ the first column of the matrix is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ the second column of the matrix is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

\Rightarrow the matrix for a reflection in $y = x$ is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Example: Find the matrix of the linear transformation, T , which maps $(1, 0, 0) \rightarrow (3, 4, 2)$,
 $(1, 1, 0) \rightarrow (6, 1, 5)$ and $(2, 1, -4) \rightarrow (1, 1, -1)$.

Solution:

Firstly $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \Rightarrow$ first column is $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

Secondly $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$ but $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \Rightarrow$ second column is $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

Thirdly $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

but $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow 2 \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - 4T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow 2 \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - 4T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow$ third column is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow T = \begin{pmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$.

Image of a line

Example: Find the image of the line $\underline{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ under \mathbf{T} ,

$$\text{where } \mathbf{T} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}.$$

Solution: As \mathbf{T} is a linear transformation, we can find

$$\mathbf{T}(\underline{r}) = \mathbf{T}\left(\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right) = \mathbf{T}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \mathbf{T}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{T}(\underline{r}) = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{T}(\underline{r}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix} \text{ and so a vector equation of the new line is}$$

$$\underline{r} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix}.$$

Image of a plane 1

Similarly the image of a plane $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$, under a linear transformation, \mathbf{T} , is

$$\mathbf{T}(\underline{r}) = \mathbf{T}(\underline{a} + \lambda \underline{b} + \mu \underline{c}) = \mathbf{T}(\underline{a}) + \lambda \mathbf{T}(\underline{b}) + \mu \mathbf{T}(\underline{c}).$$

Image of a plane 2

To find the image of a plane with equation of the form $ax + by + cz = d$, first construct a vector equation.

Method 1

Example 1: Find the image of the plane $3x - 2y + 4z = 6$ under a linear transformation, \mathbf{T} .

Solution: First construct a vector equation,

(i) Put $x = z = 0 \Rightarrow y = -3 \Rightarrow (0, -3, 0)$ is a point on the plane

(ii) To find vectors parallel to the plane, they must be perpendicular to $\underline{n} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$. By

inspection, using the top two coordinates, $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, and, using the bottom two

coordinates, $\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ must be \perp to \underline{n} (look at the scalar products), and so are parallel to the plane.

$$\Rightarrow \text{The vector equation of the plane is } \underline{\mathbf{r}} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{The image under the matrix } \mathbf{M} \text{ is } \mathbf{M}\underline{\mathbf{r}} = \mathbf{M} \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \mathbf{M} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \mathbf{M} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Example 2: Find the image of the plane $\underline{\mathbf{r}} \cdot \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} = 8$ under a linear transformation \mathbf{T} .

Solution: The equation can be written as $2x - 5y = 8$

(i) Put $y = 0 \Rightarrow x = 4 \Rightarrow (4, 0, 0)$ is a point on the plane z could be anything

(ii) To find vectors parallel to the plane, they must be perpendicular to $\underline{\mathbf{n}} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$. By

inspection, using the top two coordinates, $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, and, using the 0 z -coordinate, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

must be \perp to $\underline{\mathbf{n}}$ (look at the scalar products), and so are parallel to the plane.

Continue as in Example 1.

Method 2, as in the book

Fine until the vector $\underline{\mathbf{n}}$ has a zero coordinate, then life is a bit more complicated.

Example: Find the image of the plane $3x - 2y + 4z = 6$ under a linear transformation, \mathbf{T} .

Solution: To construct a vector equation, put $x = \lambda$, $y = \mu$ and find z in terms of λ and μ .

$$\Rightarrow 3\lambda - 2\mu + 4z = 6 \Rightarrow z = \frac{6 - 3\lambda + 2\mu}{4}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \\ \frac{6 - 3\lambda + 2\mu}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6/4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3/4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6/4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{making the numbers nicer in the 'parallel' vectors}$$

and now continue as in method 1.

NOTE that $\mathbf{M}(\underline{\mathbf{b}} \times \underline{\mathbf{c}})$ is **not** equal to $\mathbf{M}(\underline{\mathbf{b}}) \times \mathbf{M}(\underline{\mathbf{c}})$, since this does not follow the conditions of a *linear* transformation, so you must use one of the methods above.

7 Eigenvalues and eigenvectors

Definitions

- 1) An *eigenvector* of a linear transformation, T , is a non-zero vector whose *direction* is unchanged by T .

So, if \underline{e} is an eigenvector of T then its image \underline{e}' is parallel to \underline{e} , or $\underline{e}' = \lambda \underline{e}$

$$\Rightarrow \underline{e}' = T(\underline{e}) = \lambda \underline{e}.$$

\underline{e} defines a line which maps onto itself and so is invariant *as a whole line*.

If $\lambda = 1$ each point on the line remains in the same place, and we have a line of *invariant points*.

- 2) The *characteristic equation* of a matrix A is $\det(A - \lambda I) = 0$

$$A\underline{e} = \lambda \underline{e}$$

$$\Rightarrow (A - \lambda I)\underline{e} = \underline{0} \quad \text{has non-zero solutions} \quad \text{eigenvectors are non-zero}$$

$$\Rightarrow A - \lambda I \text{ is a singular matrix}$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\Rightarrow \text{the solutions of the characteristic equation are the eigenvalues.}$$

2 × 2 matrices

Example: Find the eigenvalues and eigenvectors for the transformation with matrix

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Solution: The characteristic equation is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)(4 - \lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3$$

For $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow x + y = 2x \quad \Rightarrow \quad x = y$$

$$\text{and} \quad -2x + 4y = 2y \quad \Rightarrow \quad x = y$$

$$\Rightarrow \text{eigenvector } \underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{we could use } \begin{pmatrix} 3.7 \\ 3.7 \end{pmatrix}, \text{ but why make things nasty}$$

For $\lambda_2 = 3$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow x + y = 3x \quad \Rightarrow \quad 2x = y$$

$$\text{and } -2x + 4y = 3y \quad \Rightarrow \quad 2x = y$$

$$\Rightarrow \text{eigenvector } \underline{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{choosing easy numbers.}$$

Orthogonal matrices

Normalised eigenvectors

A normalised eigenvector is an eigenvector of length 1.

In the above example, the normalized eigenvectors are $\underline{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $\underline{e}_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

Orthogonal vectors

A posh way of saying perpendicular, scalar product will be zero.

Orthogonal matrices

If the columns of a matrix form vectors which are

- (i) mutually orthogonal (or perpendicular)
- (ii) each of length 1

then the matrix is an *orthogonal* matrix.

Example:

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ are both unit vectors, and

$$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{-2}{5} + \frac{2}{5} = 0, \Rightarrow \text{the vectors are orthogonal}$$

$$\Rightarrow \mathbf{M} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \text{ is an orthogonal matrix}$$

Notice that

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so *the transpose of an orthogonal matrix is also its inverse.*

This is true for **all** orthogonal matrices think of any set of perpendicular unit vectors

Another definition of an orthogonal matrix is

$$\mathbf{M} \text{ is orthogonal} \quad \Leftrightarrow \quad \mathbf{M}^T \mathbf{M} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{M}^{-1} = \mathbf{M}^T$$

Diagonalising a 2×2 matrix

Let \mathbf{A} be a 2×2 matrix with eigenvalues λ_1 and λ_2 ,
and eigenvectors $\underline{e}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\underline{e}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$

$$\text{then } \mathbf{A} \underline{e}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_1 v_1 \end{pmatrix} \quad \text{and} \quad \mathbf{A} \underline{e}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 u_2 \\ \lambda_2 v_2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{A} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 \\ \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \dots \dots \dots \mathbf{I}$$

Define \mathbf{P} as the matrix whose columns are eigenvectors of \mathbf{A} , and \mathbf{D} as the diagonal matrix, whose entries are the eigenvalues of \mathbf{A}

$$\mathbf{I} \quad \Rightarrow \quad \mathbf{P} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{AP} = \mathbf{PD} \quad \Rightarrow \quad \mathbf{P}^{-1} \mathbf{AP} = \mathbf{D}$$

The above is the general case for diagonalising **any** matrix.

In this course we consider only diagonalising symmetric matrices.

Diagonalising 2×2 symmetric matrices

Eigenvectors of symmetric matrices

Preliminary result:

$$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\text{The scalar product } \underline{\mathbf{x}} \cdot \underline{\mathbf{y}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$$

$$\text{but } (x_1 \ x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$$

$$\Rightarrow \quad \underline{\mathbf{x}}^T \underline{\mathbf{y}} = \underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$$

This result allows us to use matrix multiplication for the scalar product.

Theorem: Eigenvectors, for different eigenvalues, of a symmetric matrix are orthogonal.

Proof: Let \mathbf{A} be a symmetric matrix, then $\mathbf{A}^T = \mathbf{A}$

$$\text{Let } \mathbf{A} \underline{\mathbf{e}}_1 = \lambda_1 \underline{\mathbf{e}}_1, \quad \text{and} \quad \mathbf{A} \underline{\mathbf{e}}_2 = \lambda_2 \underline{\mathbf{e}}_2, \quad \lambda_1 \neq \lambda_2.$$

$$\lambda_1 \underline{\mathbf{e}}_1^T = (\lambda_1 \underline{\mathbf{e}}_1)^T = (\mathbf{A} \underline{\mathbf{e}}_1)^T = \underline{\mathbf{e}}_1^T \mathbf{A}^T = \underline{\mathbf{e}}_1^T \mathbf{A} \quad \text{since } \mathbf{A}^T = \mathbf{A}, \text{ and } (\mathbf{A}\mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$$

$$\Rightarrow \quad \lambda_1 \underline{\mathbf{e}}_1^T = \underline{\mathbf{e}}_1^T \mathbf{A}$$

$$\Rightarrow \quad \lambda_1 \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2 = \underline{\mathbf{e}}_1^T \mathbf{A} \underline{\mathbf{e}}_2 = \underline{\mathbf{e}}_1^T \lambda_2 \underline{\mathbf{e}}_2 = \lambda_2 \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2$$

$$\Rightarrow \quad \lambda_1 \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2 = \lambda_2 \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2$$

$$\Rightarrow \quad (\lambda_1 - \lambda_2) \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2 = \underline{\mathbf{0}}$$

$$\text{But } \lambda_1 - \lambda_2 \neq 0 \Rightarrow \underline{\mathbf{e}}_1^T \underline{\mathbf{e}}_2 = \underline{\mathbf{0}} \Leftrightarrow \underline{\mathbf{e}}_1 \cdot \underline{\mathbf{e}}_2 = 0$$

$$\Rightarrow \quad \text{the eigenvectors are orthogonal} \quad \text{or perpendicular}$$

Diagonalising a symmetric matrix

The above theorem makes diagonalising a symmetric matrix, \mathbf{A} , easy.

- 1) Find eigenvalues, λ_1 and λ_2 , and eigenvectors, $\underline{\mathbf{e}}_1$ and $\underline{\mathbf{e}}_2$
- 2) Normalise the eigenvectors, to give $\underline{\hat{\mathbf{e}}}_1$ and $\underline{\hat{\mathbf{e}}}_2$.
- 3) Write down the matrix \mathbf{P} with $\underline{\hat{\mathbf{e}}}_1$ and $\underline{\hat{\mathbf{e}}}_2$ as columns.
 \mathbf{P} will now be an orthogonal matrix since $\underline{\hat{\mathbf{e}}}_1$ and $\underline{\hat{\mathbf{e}}}_2$ are orthogonal
 $\Rightarrow \quad \mathbf{P}^{-1} = \mathbf{P}^T$
- 4) $\mathbf{P}^T \mathbf{A} \mathbf{P}$ will be the diagonal matrix $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Example: Diagonalise the symmetric matrix $A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$.

Solution: The characteristic equation is $\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda)(9-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 15\lambda + 50 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda - 10) = 0$$

$$\Rightarrow \lambda = 5 \text{ or } 10$$

For $\lambda_1 = 5$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow 6x - 2y = 5x \quad \Rightarrow \quad x = 2y$$

$$\text{and } -2x + 9y = 5y \quad \Rightarrow \quad x = 2y$$

$$\Rightarrow \underline{\mathbf{e}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{and normalising } \Rightarrow \hat{\underline{\mathbf{e}}}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

For $\lambda_2 = 10$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 10 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow 6x - 2y = 10x \quad \Rightarrow \quad -2x = y$$

$$\text{and } -2x + 9y = 10y \quad \Rightarrow \quad -2x = y$$

$$\Rightarrow \underline{\mathbf{e}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\text{and normalising } \Rightarrow \hat{\underline{\mathbf{e}}}_2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

Notice that the eigenvectors are orthogonal

$$\Rightarrow \mathbf{P} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \mathbf{D} = \mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.$$

3 × 3 matrices

All the results for 2 × 2 matrices are also true for 3 × 3 matrices (or $n \times n$ matrices). The proofs are either the same, or similar in a higher number of dimensions.

Finding eigenvectors for 3 × 3 matrices.

Example: Given that $\lambda = 5$ is an eigenvalue of the matrix

$$M = \begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & -2 \end{pmatrix}, \text{ find the corresponding eigenvector.}$$

Solution: Consider $M\underline{e} = 5\underline{e}$

$$\Rightarrow \begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 3x - y + 2z = 5x \quad \Rightarrow \quad -2x - y + 2z = 0 \quad \mathbf{I} \\ -2x + y - z = 5y \quad \Rightarrow \quad -2x - 4y - z = 0 \quad \mathbf{II} \\ 4x - y - 2z = 5z \quad \Rightarrow \quad 4x - y - 7z = 0 \quad \mathbf{III} \end{array}$$

Now eliminate one variable, say x :

$$\mathbf{I} - \mathbf{II} \Rightarrow 3y + 3z = 0 \quad \Rightarrow \quad y = -z$$

We are not expecting to find *unique* solutions, so put $z = 1$, and then find x and y .

$$\Rightarrow y = -1, \text{ and,}$$

$$\text{from } \mathbf{I}, \quad 2x = 2z - y = 2 + 1 = 3$$

$$\Rightarrow x = 1.5$$

$$\Rightarrow \underline{e} = \begin{pmatrix} 1.5 \\ -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

as any multiple will also be an eigenvector

check in **II** and **III**, **O.K.**

Diagonalising 3×3 symmetric matrices

Example: $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

Find an orthogonal matrix P such that $P^T A P$ is a diagonal matrix.

Solution:

1) Find eigenvalues

The characteristic equation is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[- \lambda(1-\lambda) - 4] + 2 \times [2\lambda - 0] + 0 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$$

By inspection $\lambda = -2$ is a root $\Rightarrow (\lambda + 2)$ is a factor

$$\Rightarrow (\lambda + 2)(\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow (\lambda + 2)(\lambda - 1)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = -2, 1 \text{ or } 4.$$

2) Find normalized eigenvectors

$$\lambda_1 = -2 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} \Rightarrow 2x - 2y &= -2x & \text{I} \\ -2x + y + 2z &= -2y & \text{II} \\ 2y &= -2z & \text{III} \end{aligned}$$

$$\text{I} \Rightarrow y = 2x, \quad \text{and III} \Rightarrow y = -z \quad \text{choose } x = 1 \text{ and find } y \text{ and } z$$

$$\Rightarrow \mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad |\mathbf{e}_1| = e_1 = \sqrt{9} = 3 \Rightarrow \hat{\mathbf{e}}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$\lambda_2 = 1 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} \Rightarrow 2x - 2y &= x & \text{I} \\ -2x + y + 2z &= y & \text{II} \\ 2y &= z & \text{III} \end{aligned}$$

$$\mathbf{I} \Rightarrow x = 2y, \text{ and } \mathbf{II} \Rightarrow z = 2y \quad \text{choose } y = 1 \text{ and find } x \text{ and } z$$

$$\Rightarrow \underline{\mathbf{e}}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad |\underline{\mathbf{e}}_2| = e_2 = \sqrt{9} = 3$$

$$\Rightarrow \hat{\underline{\mathbf{e}}}_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$\lambda_3 = 4 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad 2x - 2y &= 4x && \mathbf{I} \\ -2x + y + 2z &= 4y && \mathbf{II} \\ 2y &= 4z && \mathbf{III} \end{aligned}$$

$$\mathbf{I} \Rightarrow x = -y, \text{ and } \mathbf{III} \Rightarrow y = 2z \quad \text{choose } z = 1 \text{ and find } x \text{ and } y$$

$$\Rightarrow \underline{\mathbf{e}}_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad |\underline{\mathbf{e}}_3| = e_3 = \sqrt{9} = 3$$

$$\Rightarrow \hat{\underline{\mathbf{e}}}_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

3) Find orthogonal matrix, P

$$\Rightarrow \mathbf{P} = (\hat{\underline{\mathbf{e}}}_1 \quad \hat{\underline{\mathbf{e}}}_2 \quad \hat{\underline{\mathbf{e}}}_3)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$$

is required orthogonal matrix

4) Find diagonal matrix, D

$$\Rightarrow \mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

A nice long question! But, although you will not be asked to do a complete problem, the examiners can test every step above!

Index

Differentiation

- hyperbolic functions, 10
- inverse hyperbolic functions, 10

Further coordinate systems

- hyperbola, 7
- locus problems, 9
- parabola, 7
- parametric differentiation, 8
- tangents and normals, 8

Hyperbolic functions

- addition formulae, 3
- definitions and graphs, 3
- double angle formulae, 3
- inverse functions, 4
- inverse functions, logarithmic form, 4
- Osborne's rule, 3
- solving equations, 5

Integration

- area of a surface, 20
- inverse hyperbolic functions, 14
- inverse trig functions, 14
- $\ln x$, 14
- reduction formulae, 15
- standard techniques, 12

Matrices

- cofactors matrix, 40
- determinant of 3×3 matrix, 39
- diagonalising 2×2 matrices, 48
- diagonalising symmetric 2×2 matrices, 49
- diagonalising symmetric 3×3 matrices, 52
- dimension, 39
- identity matrix, 39
- inverse of 3×3 matrix, 40
- non-singular matrix, 40
- singular matrix, 40
- symmetric matrix, 39
- transpose matrix, 39
- zero matrix, 39

Matrices - eigenvalues and eigenvectors

- 2×2 matrices, 46
- characteristic equation, 46
- for 3×3 matrix, 51
- normalised eigenvectors, 47
- orthogonal matrices, 47
- orthogonal vectors, 47

Matrices - linear transformations

- base vectors, 42
- image of a line, 44
- image of a plane, 44

Vectors

- area of triangle, 23
- triple scalar product, 24
- vector product, 22
- volume of parallelepiped, 24
- volume of tetrahedron, 25

Vectors - lines

- angle between line and plane, 37
- cartesian equation of line in 3-D, 26
- distance between skew lines, 34
- distance from point to line, 31, 35
- line of intersection of two planes, 36
- vector equation of a line, 25
- vector product equation of a line, 26

Vectors - planes

- angle between line and plane, 37
- angle between two planes, 38
- Cartesian equation, 27
- distance between parallel planes, 31, 33
- distance of a point from a plane, 29, 32
- line of intersection of two planes, 36
- reflection of point in plane, 30
- vector equation, 28