Pure Core 4

Revision Notes

June 2016

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1 Algebra

Partial fractions

1) You must start with a proper fraction: i.e. the degree of the numerator **must be less than** the degree of the denominator.

   If this is not the case you must **first** do long division to find quotient and remainder.

2) (*a*) Linear factors (not repeated)

$$\frac{\ldots\ldots}{(ax-b)(\ldots\ldots)} \equiv \frac{A}{(ax-b)} + \;\ldots\ldots\ldots$$

   (*b*) Linear repeated factors (squared)

$$\frac{\ldots\ldots}{(ax-b)^2(\ldots\ldots)} \equiv \frac{A}{(ax-b)^2} + \frac{B}{(ax-b)} + \;\ldots\ldots\ldots$$

   (*c*) Quadratic factors)

$$\frac{\ldots\ldots}{(ax^2+b)(\ldots\ldots)} \equiv \frac{Ax+B}{(ax^2+b)} + \;\ldots\ldots\ldots$$

   or

$$\frac{\ldots\ldots}{(ax^2+bx+c)(\ldots\ldots)} \equiv \frac{Ax+B}{(ax^2+bx+c)} + \;\ldots\ldots\ldots$$

Example: Express $\frac{5-x+2x^2}{(1-x)(1+x^2)}$ in partial fractions.

Solution: The degree of the numerator, 2, is less than the degree of the denominator, 3, so we do not need long division and can write

$$\frac{5-x+2x^2}{(1-x)(1+x^2)} \equiv \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$$ multiply both sides by $(1-x)(1+x^2)$

$\Rightarrow \quad 5-x+2x^2 \equiv A(1+x^2) + (Bx+C)(1-x)$

$\Rightarrow \quad 5-1+2 = 2A \quad \Rightarrow \quad A = 3$ — clever value!, put $x = 1$

$\Rightarrow \quad 5 = A + C \quad \Rightarrow \quad C = 2$ — easy value, put $x = 0$

$\Rightarrow \quad 2 = A - B \quad \Rightarrow \quad B = 1$ — equate coefficients of x^2

$$\Rightarrow \quad \frac{5-x+2x^2}{(1-x)(1+x^2)} \equiv \frac{3}{1-x} + \frac{x+2}{1+x^2}.$$

Note: You can put in any value for x, so you can always find as many equations as you need to solve for $A, B, C, D \ldots$.

Example: Express $\dfrac{x^2-7x+22}{(2x-1)(x-3)^2}$ in partial fractions.

Solution: The degree of the numerator, 2, is less than the degree of the denominator, 3, so we do not need long division and can write

$$\frac{x^2-7x+22}{(2x-1)(x-3)^2} \equiv \frac{A}{2x-1} + \frac{B}{(x-3)^2} + \frac{C}{x-3} \qquad \text{multiply by denominator}$$

$\Rightarrow \quad x^2 - 7x + 22 \equiv A(x-3)^2 + B(2x-1) + C(2x-1)(x-3)$

$\Rightarrow \quad 9 - 21 + 22 = 5B \quad \Rightarrow \quad B = 2$ — clever value, put $x = 3$

$\Rightarrow \quad \frac{1}{4} - \frac{7}{2} + 22 = \left(\frac{5}{2}\right)^2 A \quad \Rightarrow \quad A = 3$ — clever value, put $x = ½$

$\Rightarrow \quad 22 = 9A - B + 3C \quad \Rightarrow \quad C = -1$ — easy value, put $x = 0$

$$\Rightarrow \quad \frac{x^2-7x+22}{(2x-1)(x-3)^2} \equiv \frac{3}{2x-1} + \frac{2}{(x-3)^2} - \frac{1}{x-3}$$

Example: Express $\dfrac{x^3+x^2-9x-3}{x^2-9}$ in partial fractions.

Solution: Firstly the degree of the numerator is **not less** than the degree of the denominator so we must divide top by bottom.

```
x² – 9 ) x³  +  x²  –  9x  –  3   ( x + 1
         x³         –  9x
                x²          –  3
                x²          –  9
                               6
```

$$\Rightarrow \quad \frac{x^3+x^2-9x-3}{x^2-9} = x + 1 + \frac{6}{x^2-9}$$

Factorise to give $x^2 - 9 = (x-3)(x+3)$ and write

$$\frac{6}{x^2-9} \equiv \frac{6}{(x-3)(x+3)} \equiv \frac{A}{x-3} + \frac{B}{x+3} \qquad \text{multiplying by denominator}$$

$\Rightarrow \quad 6 \equiv A(x+3) + B(x-3)$

$\Rightarrow \quad 6 = 6A \quad \Rightarrow \quad A = 1$ — clever value, put $x = 3$

$\Rightarrow \quad 6 = -6B \quad \Rightarrow \quad B = -1$ — clever value, put $x = -3$

$$\Rightarrow \quad \frac{x^3+x^2-9x-3}{x^2-9} = x + 1 + \frac{1}{x-3} - \frac{1}{x+3}$$

2 Coordinate Geometry

Parametric equations

If we define x and y in terms of a single variable (the letters t or θ are often used) then this variable is called a parameter: we then have the parametric equation of a curve.

Example: $x = 2 + t$, $y = t^2 - 3$ is the parametric equation of a curve. Find

(i) the points where the curve meets the x-axis,

(ii) the points of intersection of the curve with the line $y = 2x + 1$.

Solution:

(i) The curve meets the x-axis when $y = 0 \Rightarrow t^2 = 3 \Rightarrow t = \pm\sqrt{3}$

$\Rightarrow$ curve meets the x-axis at $(2 - \sqrt{3}, 0)$ and $(2 + \sqrt{3}, 0)$.

(ii) Substitute for x and y in the equation of the line

$y = 2x + 1$, and $y = t^2 - 3$, $x = 2 + t$

$\Rightarrow$ $t^2 - 3 = 2(2 + t) + 1$

$\Rightarrow$ $t^2 - 2t - 8 = 0 \Rightarrow (t - 4)(t + 2) = 0$

$\Rightarrow$ $t = 4$ or -2

$\Rightarrow$ the points of intersection are $(6, 13)$ and $(0, 1)$.

Example: Find whether the curves $x = 2t + 3$, $y = t^2 - 2$ and $x = s - 1$, $y = s - 3$ intersect. If they do give the point of intersection, otherwise give reasons why they do not intersect.

Solution: If they intersect there must be values of t and s (not necessarily the same), which make their ***x***-coordinates equal, so for these values of t and s

$\Rightarrow$ $2t + 3 = s - 1 \Rightarrow s = 2t + 4$

The ***y***-coordinates must also be equal for the same values of t and s

$\Rightarrow$ $t^2 - 2 = s - 3 = (2t + 4) - 3 = 2t + 1$ since $s = 2t + 4$

$\Rightarrow$ $t^2 - 2t - 3 = 0 \Rightarrow (t - 3)(t + 1) = 0$

$\Rightarrow$ $t = 3$, s = 10 or $t = -1$, $s = 2$

$\Rightarrow$ Curves intersect at $t = 3$ giving $(9, 7)$. Check $s = 10$ gives $(9, 7)$.

Or curves intersect at $t = -1$ giving $(1, -1)$. Check $s = 2$ giving $(1, -1)$.

Conversion from parametric to Cartesian form

Eliminate the parameter (t or θ or …) to form an equation between x and y **only**.

Example: Find the Cartesian equation of the curve given by $y = t^2 - 3,\ x = t + 2$.

Solution: $x = t + 2 \quad \Rightarrow \quad t = x - 2$, and $y = t^2 - 3$

$\Rightarrow \quad y = (x - 2)^2 - 3,$

which is the Cartesian equation of a parabola with vertex at (2, –3)

With trigonometric parametric equations the formulae

$$\sin^2 t + \cos^2 t = 1 \quad \text{and} \quad \sec^2 t - \tan^2 t = 1$$

will often be useful.

Example: Find the Cartesian equation of the curve given by

$y = 3\sin t + 2,\ x = 3\cos t - 1.$

Solution: Re-arranging we have

$\sin t = \dfrac{y-2}{3}$, and $\cos t = \dfrac{x+1}{3}$, which together with $\sin^2 t + \cos^2 t = 1$

$$\Rightarrow \quad \left(\frac{y-2}{3}\right)^2 + \left(\frac{x+1}{3}\right)^2 = 1$$

$$\Rightarrow \quad (x+1)^2 + (y-2)^2 = 9$$

which is the Cartesian equation of a circle with centre (–1, 2) and radius 3.

Example: Find the Cartesian equation of the curve given by $y = 3\tan t,\ x = 4\sec t$.

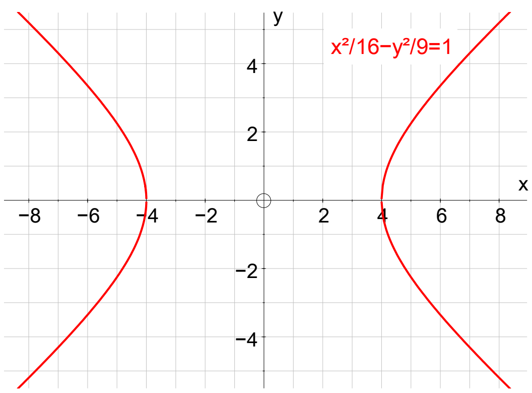
Hence sketch the curve.

Solution: Re-arranging we have

$\tan t = \dfrac{y}{3}$, and $\sec t = \dfrac{x}{4}$,

which together with $\sec^2 t - \tan^2 t = 1$

$$\Rightarrow \quad \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$



which is the standard equation of a hyperbola with centre (0, 0)

and x-intercepts (4, 0), (–4, 0).

Area under curve given parametrically

We know that the area between a curve and the x-axis is given by $A = \int y\, dx$

$\Rightarrow \quad \dfrac{dA}{dx} = y$.

But, from the chain rule $\quad \dfrac{dA}{dt} = \dfrac{dA}{dx} \times \dfrac{dx}{dt} \quad \Rightarrow \quad \dfrac{dA}{dt} = y\,\dfrac{dx}{dt}$

Integrating with respect to t

$\Rightarrow \quad A = \int y \dfrac{dx}{dt}\, dt$.

Example: Find the area between the curve $y = t^2 - 1$, $x = t^3 + t$, the x-axis and the lines $x = 0$ and $x = 2$.

Solution: The area is $A = \int_0^2 y\ dx$.

$\Rightarrow \quad A = \int_?^? y \dfrac{dx}{dt}\ dt$ $\qquad$ we must write limits for t, **not** x

Firstly we need to find y and $\dfrac{dx}{dt}$ in terms of t.

$y = t^2 - 1$ and $\dfrac{dx}{dt} = 3t^2 + 1$.

Secondly we are integrating with respect to t and so the limits of integration must be for values of t.

$x = 0 \;\Rightarrow\; t = 0$, and

$x = 2 \;\Rightarrow\; t^3 + t = 2 \;\Rightarrow\; t^3 + t - 2 = 0 \;\Rightarrow\; (t - 1)(t^2 + t + 2) = 0 \;\Rightarrow\; t = 1$ only.

so the limits for t are from 0 to 1

$\Rightarrow \quad A = \int_0^1 y \dfrac{dx}{dt}\, dt = \int_0^1 (t^2 - 1)\,(3t^2 + 1)\, dt$

$= \int_0^1 3t^4 - 2t^2 - 1\ dt$

$= \left[\dfrac{3t^5}{5} - \dfrac{2t^3}{3} - t\right]_0^1 = -1\tfrac{1}{15}$

Note that in simple problems you may be able to eliminate t and find $\int y\, dx$ in the usual manner. However there will be some problems where this is difficult and the above technique will be better.

3 Sequences and series

Binomial series $(1 + x)^n$ for any n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \times x^2 + \frac{n(n-1)(n-2)}{3!} \times x^3 + \ldots.$$

This converges provided that $|x| < 1$.

Example: Expand $(1 + 3x)^{-2}$, giving the first four terms, and state the values of x for which the series is convergent.

Solution:

$$(1+3x)^{-2} = 1 + (-2) \times 3x + \frac{(-2)\times(-3)}{2!} \times (3x)^2 + \frac{(-2)\times(-3)\times(-4)}{3!} \times (3x)^3$$

$$= 1 - 6x + 27x^2 - 108x^3 + \ldots$$

This series is convergent when $|3x| < 1 \iff |x| < {}^1/_3$.

Example: Use the previous example to find an approximation for $\frac{1}{0\cdot9997^2}$.

Solution: Notice that $\frac{1}{0\cdot9997^2} = 0\cdot9997^{-2} = (1 + 3x)^{-2}$ when $x = -0\cdot0001$.

So writing $x = -0\cdot0001$ in the expansion $1 - 6x + 27x^2 - 108x^3$

$0\cdot9997^{-2} \approx 1 + 0\cdot0006 + 0\cdot00000027 + 0\cdot000000000108 = 1\cdot000600270108$

The correct answer to 13 decimal places is $1\cdot0006002701080$ not bad eh?

Example: Expand $(4 - x)^{\frac{1}{2}}$, giving all terms up to and including the term in x^3, and state for what values of x the series is convergent.

Solution: As the formula holds for $(\mathbf{1} + x)^n$ we first re-write

$$(4-x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1 - \frac{x}{4}\right)^{\frac{1}{2}} = 2 \times \left(1 - \frac{x}{4}\right)^{\frac{1}{2}}$$ and now we can use the formula

$$= 2 \times \left(1 + \frac{1}{2}\left(-\frac{x}{4}\right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!} \times \left(-\frac{x}{4}\right)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!} \times \left(-\frac{x}{4}\right)^3 + \ldots\right)..$$

$$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512}.$$

This expansion converges for $\left|\frac{x}{4}\right| < 1 \iff |x| < 4$.

Example: Find the expansion of $\dfrac{3x-1}{x^2-x-6}$ in ascending powers of x up to x^2.

Solution: First write in partial fractions

$$\Rightarrow \quad \frac{3x-1}{x^2-x-6} = \frac{2}{x+3} + \frac{1}{x-2}, \quad \text{which must now be written as}$$

$$\frac{2}{3(1+\frac{x}{3})} - \frac{1}{2(1-\frac{x}{2})} = \tfrac{2}{3}(1+\tfrac{x}{3})^{-1} - \tfrac{1}{2}(1-\tfrac{x}{2})^{-1}$$

$$= \frac{2}{3}\left(1 + (-1)\left(\frac{x}{3}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{3}\right)^2\right) - \frac{1}{2}\left(1 + (-1)\left(\frac{-x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-x}{2}\right)^2\right)$$

$$= \frac{1}{6} - \frac{17x}{36} - \frac{11x^2}{216}.$$

4 Differentiation

Relationship between $\frac{dy}{dx}$ and $\frac{dx}{dy}$

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{using the chain rule}$$

So if $y = 3x^2 \Rightarrow \frac{dy}{dx} = 6x \Rightarrow \frac{dx}{dy} = \frac{1}{6x}$.

Implicit differentiation

This is just the chain rule when we do not know *explicitly* what y is as a function of x.

Examples: The following examples use the chain rule (or implicit differentiation)

$$\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy} \times \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{d(\sin y)}{dx} = \frac{d(\sin y)}{dy} \times \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$\frac{d(5x^2y)}{dx} = 10xy + 5x^2\frac{dy}{dx} \qquad \text{using the product rule}$$

$$\frac{d}{dx}\left(x^2+3y\right)^3 = 3\left(x^2+3y\right)^2 \times \frac{d}{dx}\left(x^2+3y\right) = 3\left(x^2+3y\right)^2 \times \left(2x+3\frac{dy}{dx}\right)$$

Example: Find the gradient of, and the equation of, the tangent to the curve $x^2 + y^2 - 3xy = -1$ at the point $(1, 2)$.

Solution: Differentiating $x^2 + y^2 - 3xy = -1$ with respect to x gives

$$2x + 2y\frac{dy}{dx} - \left(3y + 3x\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3y-2x}{2y-3x} \quad \Rightarrow \quad \frac{dy}{dx} = 4 \quad \text{when } x = 1 \text{ and } y = 2.$$

Equation of the tangent is $y - 2 = 4(x - 1)$

$\Rightarrow \quad y = 4x - 2$.

Parametric differentiation

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example: A curve has parametric equations $x = t^2 + t$, $y = t^3 - 3t$.

(i) Find the equation of the normal at the point where $t = 2$.

(ii) Find the points with zero gradient.

Solution:

(i) When $t = 2$, $x = 6$ and $y = 2$.

$$\frac{dy}{dt} = 3t^2 - 3 \text{ and } \frac{dx}{dt} = 2t + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t + 1} = \frac{9}{5} \quad \text{when } t = 2$$

Thus the gradient of the normal at the point (6, 2) is $\frac{-5}{9}$

and its equation is $y - 2 = \frac{-5}{9}(x - 6)$ $\Rightarrow$ $5x + 9y = 48$.

(ii) gradient = 0 when $\frac{dy}{dx} = \frac{3t^2 - 3}{2t + 1} = 0$

$\Rightarrow$ $3t^2 - 3 = 0$

$\Rightarrow$ $t = \pm 1$

$\Rightarrow$ points with zero gradient are (0, 2) and (2, –2).

Exponential functions, a^x

Proof (i) $y = a^x$

$\Rightarrow$ $\ln y = \ln a^x = x \ln a$

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = \ln a$ $\Rightarrow$ $\frac{dy}{dx} = y \ln a$

$\Rightarrow$ $\frac{d(a^x)}{dx} = a^x \ln a$

Proof (ii) $y = a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$ since $a = e^{\ln a}$

$\Rightarrow$ $\frac{dy}{dx} = e^{x \ln a} \times \ln a = a^x \ln a$ chain rule

$\Rightarrow$ $\frac{d(a^x)}{dx} = a^x \ln a$

Example: Find the derivative of $y = 3^{x^2}$.

Solution: $\dfrac{dy}{dx} = 3^{x^2} \ln 3 \times \dfrac{d(x^2)}{dx} = 3^{x^2} \ln 3 \times 2x$

Example: Find the derivative of $y = 5^{\sin x}$.

Solution: $\dfrac{dy}{dx} = 5^{\sin x} \ln 5 \times \dfrac{d(\sin x)}{dx} = 5^{\sin x} \ln 5 \times \cos x$

Related rates of change

We can use the chain rule to relate one rate of change to another.

Example: A spherical snowball is melting at a rate of 96 $cm^3\ s^{-1}$ when its radius is 12 cm.

Find the rate at which its surface area is decreasing at that moment.

Solution: We know that $V = \frac{4}{3}\pi r^3$ and that $A = 4\pi r^2$.

Using the chain rule we have

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}, \qquad \text{since } \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\Rightarrow \quad 96 = 4 \times \pi \times 12^2 \times \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{6\pi} \quad cm\ s^{-1}$$

Using the chain rule again

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times \frac{dr}{dt}, \qquad \text{since } \frac{dA}{dr} = 8\pi r$$

$$\Rightarrow \quad \frac{dA}{dt} = 8 \times \pi \times 12 \times \frac{1}{6\pi} = 16\ cm^2\ s^{-1}$$

Forming differential equations

Example: The mass of a radio-active substance at time t is decaying at a rate which is proportional to the mass present at time t. Find a differential equation connecting the mass m and the time t.

Solution: Remember that $\frac{dm}{dt}$ means the rate at which the mass is **increasing** so in this case we must consider the rate of decay as a negative increase

$$\Rightarrow \quad \frac{dm}{dt} \propto -m$$

$$\Rightarrow \quad \frac{dm}{dt} = -km,$$ where k is the (*positive*) constant of proportionality.

$$\Rightarrow \quad \int \frac{1}{m}\, dm = \int -kt\ dt$$

$$\Rightarrow \quad \ln|m| = -\tfrac{1}{2}kt^2 + \ln A$$

$$\Rightarrow \quad \ln\left|\frac{m}{A}\right| = -\frac{1}{2}kt^2$$

$$\Rightarrow \quad m = Ae^{-\frac{1}{2}kt^2}$$

5 Integration

Integrals of e^x and $\frac{1}{x}$

$$\int e^x \ dx = e^x + c$$

$$\int \frac{1}{x} \ dx = \ln|x| + c$$ for a further treatment of this result, see the appendix

Example: Find $\int \frac{x^3 + 3x}{x^2} \ dx$

Solution: $\int \frac{x^3 + 3x}{x^2} \ dx = \int x + \frac{3}{x} \ dx = ½\, x^2 + 3 \ln |x| + c.$

Standard integrals

x must be in RADIANS when integrating trigonometric functions.

$f(x)$	$\int f(x) \ dx$	$f(x)$	$\int f(x) \ dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\sin x$	$-\cos x$
$\frac{1}{x}$	$\ln |x|$	$\cos x$	$\sin x$
e^x	e^x	$\sec x \tan x$	$\sec x$
		$\sec^2 x$	$\tan x$
		$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
		$\operatorname{cosec}^2 x$	$-\cot x$

Integration using trigonometric identities

Example: Find $\int \cot^2 x \ dx$.

Solution: $\cot^2 x = \operatorname{cosec}^2 x - 1$

$\Rightarrow \quad \int \cot^2 x \ dx = \int \operatorname{cosec}^2 x - 1 \ dx$

$= \quad -\cot x - x + c.$

Example: Find $\int \sin^2 x \; dx$.

Solution: $\sin^2 x = ½ (1 - \cos 2x)$

$\Rightarrow \quad \int \sin^2 x \; dx = \int ½ (1 - \cos 2x) \; dx$

$= \quad ½ x - ¼ \sin 2x + c$.

You **cannot** change x to $3x$ in the above result to find $\int \sin^2 3x \; dx$. see next example

Example: Find $\int \sin^2 3x \; dx$.

Solution: $\sin^2 3x = ½ (1 - \cos 2 \times 3x) = ½ (1 - \cos 6x)$

$\Rightarrow \quad \int \sin^2 3x \; dx = \int ½ (1 - \cos 6x) \; dx$

$= \quad ½ x - {}^1/_{12} \sin 6x + c$.

Example: Find $\int \sin 3x \cos 5x \; dx$.

Solution: Using the formula $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

This formula is NOT in the formula booklet – you can use the formulae for sin(A ± B) and add them

$\int \sin 3x \cos 5x \; dx$

$= \quad {}^1/_2 \int \sin 8x + \sin(-2x) \; dx \quad = \quad {}^1/_2 \int \sin 8x - \sin 2x \; dx$

$= \quad - {}^1/_{16} \cos 8x + ¼ \cos 2x + c$.

Integration by 'reverse chain rule'

Some integrals which are not standard functions can be integrated by thinking of the chain rule for differentiation.

Example: Find $\int \sin^4 3x \cos 3x \; dx$.

Solution: $\int \sin^4 3x \cos 3x \; dx$

If we think of $u = \sin 3x$, then the integrand looks like $u^4 \dfrac{du}{dx}$ if we ignore the constants, which would integrate to give ${}^1/_5 \, u^5$

so we differentiate $u^5 = \sin^5 3x$

to give $\dfrac{d}{dx}\left(\sin^5 3x\right) = 5\left(\sin^4 3x\right) \times 3 \cos 3x = 15 \sin^4 3x \cos 3x$

which is 15 times what we want and so

$\int \sin^4 3x \cos 3x \; dx = \frac{1}{15} \sin^5 3x + c$

Example: Find $\int \frac{x}{(2x^2-3)}\,dx$

Solution: $\int \frac{x}{(2x^2-3)}\,dx$

If we think of $u = (2x^2 - 3)$, then the integrand looks like $\frac{1}{u}\frac{du}{dx}$ if we ignore the constants, which would integrate to $\ln|u|$

so we differentiate $\ln|u| = \ln|2x^2 - 3|$

to give $\frac{d}{dx}\left(\ln|2x^2-3|\right) = \frac{1}{2x^2-3} \times 4x = \frac{4x}{(2x^2-3)}$

which is 4 times what we want and so

$\int \frac{x}{(2x^2-3)}\,dx = ¼ \ln|2x^2-3| + c.$

In general $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

Example: Find $\int xe^{2x^2}\,dx$

Solution: First consider $\frac{d}{dx}\left(e^{2x^2}\right) = 4x\,e^{2x^2}$, which is 4 × the integrand

$\Rightarrow \int xe^{2x^2}\,dx = \frac{e^{2x^2}}{4} + c$

Example: Find $\int 5^{3x}\,dx$

Solution: We know that $\frac{d}{dx}(5^{3x}) = 5^{3x}\ln 5 \times 3,$ using the chain rule

$\Rightarrow \int 5^{3x}\,dx = \frac{5^{3x}}{3\ln 5} + c$

Integrals of tan *x* and cot *x*

$\int \tan x\;dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}\,dx,$

and we now have $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\Rightarrow \int \tan x\;dx = -\ln|\cos x| + c$

$\Rightarrow \int \tan x\;dx = \ln|\sec x| + c$

cot *x* can be integrated by a similar method to give

$\int \cot x\;dx = \ln|\sin x| + c$

Integrals of sec *x* and cosec *x*

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

The top is now the derivative of the bottom

and we have $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$

$\Rightarrow \quad \int \sec x\, dx = \ln|\sec x + \tan x| + c$

and similarly

$$\int \operatorname{cosec} x\, dx = -\ln|\operatorname{cosec} x + \cot x| + c$$

Integration using partial fractions

For use with algebraic fractions where the denominator factorises.

Example: Find $\int \frac{6x}{x^2 + x - 2}\, dx$

Solution: First express $\frac{6x}{x^2 + x - 2}$ in partial fractions.

$$\frac{6x}{x^2 + x - 2} \equiv \frac{6x}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$$

$\Rightarrow \quad 6x \equiv A(x + 2) + B(x - 1).$

put $x = 1 \quad \Rightarrow \quad A = 2,$

put $x = -2 \quad \Rightarrow \quad B = 4$

$\Rightarrow \quad \int \frac{6x}{x^2 + x - 2}\, dx = \int \frac{2}{x-1} + \frac{4}{x+2}\, dx$

$= \quad 2 \ln|x - 1| + 4 \ln|x + 2| + c.$

Integration by substitution, indefinite

(i) Use the given substitution involving x and u, (or find a suitable substitution).

(ii) Find **either** $\frac{du}{dx}$ **or** $\frac{dx}{du}$, whichever is easier and re-arrange to find dx in terms of du, i.e $dx = \ldots\ldots du$

(iii) Use the substitution in (i) to make the integrand a function of u, and use your answer to (ii) to replace dx by ...... du.

(iv) Simplify and integrate the function of u.

(v) Use the substitution in (i) to write your answer in terms of x.

Example: Find $\int x\sqrt{3x^2-5}\ dx$ using the substitution $u = 3x^2 - 5$.

Solution: (i) $u = 3x^2 - 5$

(ii) $\frac{du}{dx} = 6x \quad \Rightarrow \quad dx = \frac{du}{6x}$

(iii) We can see that there an x will cancel, and $\sqrt{3x^2-5} = \sqrt{u}$

$$\int x\sqrt{3x^2-5}\ dx = \int x\sqrt{u}\ \frac{du}{6x} = \int \frac{\sqrt{u}}{6}\ du$$

(iv) $= \frac{1}{6}\int u^{\frac{1}{2}}\ du = \frac{1}{6} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

(v) $= \frac{(3x^2-5)^{\frac{3}{2}}}{9} + c$

Example: Find $\int \frac{1}{1+x^2}\ dx$ using the substitution $x = \tan u$.

Solution: (i) $x = \tan u$.

(ii) $\frac{dx}{du} = \sec^2 u \quad \Rightarrow \quad dx = \sec^2 u\ du.$

(iii) $\int \frac{1}{1+x^2}\ dx = \int \frac{1}{1+\tan^2 u} \sec^2 u\ du$

(iv) $= \int \frac{\sec^2 u}{\sec^2 u}\ du$ since $1 + \tan^2 u = \sec^2 u$

$= \int du = u + c$

(v) $= \tan^{-1} x + c.$

Example: Find $\int \frac{3x}{\sqrt{x^2-4}}\ dx$ using the substitution $u^2 = x^2 - 4$.

Solution: (i) $u^2 = x^2 - 4$.

(ii) **Do not re-arrange as** $u = \sqrt{x^2-4}$

We know that $\frac{d}{dx}\left(u^2\right) = 2u\frac{du}{dx}$ so differentiating gives

$$2u\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{u}{x}\ du.$$

(iii) We can see that an x will cancel and $\sqrt{x^2 - 4} = u$ so

$$\int \frac{3x}{\sqrt{x^2-4}}\,dx = \int \frac{3x}{u} \times \frac{u}{x}\,du$$

(iv) $= \int 3\,du = 3u + c$

(v) $= 3\sqrt{x^2 - 4} + c$

A justification of this technique is given in the appendix.

Integration by substitution, definite

If the integral has limits then proceed as before but remember to change the limits from values of x to the corresponding values of u.

Add (ii) (a) Change limits from x to u, and

new (v) Put in limits for u.

Example: Find $\int_2^6 x\sqrt{3x-2}\,dx$ using the substitution $u = 3x - 2$.

Solution: (i) $u = 3x - 2$.

(ii) $\frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

(ii) (a) Change limits from x to u

$x = 2 \Rightarrow u = 3 \times 2 - 2 = 4$, and $x = 6 \Rightarrow u = 3 \times 6 - 2 = 16$

(iii) $\int_2^6 x\sqrt{3x-2}\,dx = \int_4^{16} \frac{u+2}{3} \times u^{\frac{1}{2}}\ \frac{du}{3}$

(iv) $= \frac{1}{9}\int_4^{16} u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\ du$

$= \frac{1}{9}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + 2 \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^{16}$

(v) $= \frac{1}{9}\left[\frac{2}{5} \times 1024 + \frac{4}{3} \times 64\right] - \frac{1}{9}\left[\frac{2}{5} \times 32 + \frac{4}{3} \times 8\right] = 52{\cdot}4$ to 3 S.F.

Choosing the substitution

In general put u equal to the 'awkward bit' – but there are some special cases where this will not help.

$\int x^3(x^2+1)^5 \; dx$ put $u = x^2 + 1$

$\int \frac{3x}{(x-2)^2} \; dx$ put $u = x - 2$

$\int x\sqrt{2x+5} \; dx$ put $u = 2x + 5$ or $u^2 = 2x + 5$

$\int x^n \sqrt{4-x^2} \; dx$ or $\int \frac{x^n}{\sqrt{4-x^2}} \; dx$

put u or $u^2 = 4 - x^2$ **only if *n* is ODD**

put $x = 2 \sin u$ **only if *n* is EVEN (or zero)**

this makes $\sqrt{4-x^2} = \sqrt{4\cos^2 u} = 2 \cos u$

There are many more possibilities – use your imagination!!

Example: Find $I = \int \sqrt{16-x^2} \; dx$, and express your answer in as simple a form as possible.

Solution: This is of the form $\int x^n \sqrt{16-x^2} \; dx$ where $n = 0$, an even number

$\Rightarrow$ use the substitution $x = 4 \sin u$,

$\Rightarrow$ $dx = 4 \cos u \, du$

$\Rightarrow$ $I = \int \sqrt{16 - 16\sin^2 u} \times 4\cos u \; du$

$= \int 16\cos^2 u \; du = 8\int 1 + \cos 2u \; du$

$= 8\left(u + \frac{1}{2}\sin 2u\right) + c$

$x = 4 \sin u \Rightarrow u = \arcsin\left(\frac{x}{4}\right)$,

but $\sin 2u = \sin\left(2\arcsin\left(\frac{x}{4}\right)\right)$ is **not** in the simplest form.

Instead write $I = 8\left(u + \frac{1}{2} \times 2\sin u \cos u\right) + c$,

use $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-\left(\frac{x}{4}\right)^2}$

$\Rightarrow$ $I = 8\arcsin\left(\frac{x}{4}\right) + 8 \times \frac{x}{4} \times \sqrt{1-\left(\frac{x}{4}\right)^2} + c$

$= 8\arcsin\left(\frac{x}{4}\right) + \frac{x}{2}\sqrt{16-x^2} + c$

Integration by parts

The product rule for differentiation is

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \Rightarrow \quad u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}$$

$$\Rightarrow \quad \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

To integrate by parts

(i) choose u and $\frac{dv}{dx}$

(ii) find v and $\frac{du}{dx}$

(iii) substitute in formula and integrate.

Example: Find $\int x\sin x\ dx$

Solution: (i) Choose $u = x$, because it disappears when differentiated

and choose $\frac{dv}{dx} = \sin x$

(ii) $u = x \Rightarrow \frac{du}{dx} = 1$ and

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

(iii) $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$\Rightarrow \quad \int x\sin x\ dx = -x\cos x - \int 1\times(-\cos x)\ dx$

$= \quad -x\cos x + \sin x + c.$

Example: Find $\int \ln x \; dx$

Solution: (i) It does not look like a product, $u\dfrac{dv}{dx}$, but if we take $u = \ln x$ and

$\dfrac{dv}{dx} = 1$ then $u\dfrac{dv}{dx} = \ln x \times 1 = \ln x$

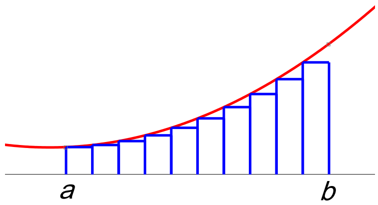
(ii) $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 1 \Rightarrow v = x$

(iii) $\int \ln x \times 1 \; dx = x \ln x - \int x \times \dfrac{1}{x} \; dx$

$= x \ln x - x + c.$

Area under curve

We found in Core 2 that the area under the curve is written as the integral $\int_a^b y \; dx$.



We can consider the area as approximately the sum of the rectangles shown.

If each rectangle has width δx and if the heights of the rectangles are $y_1, y_2, \ldots, y_n$

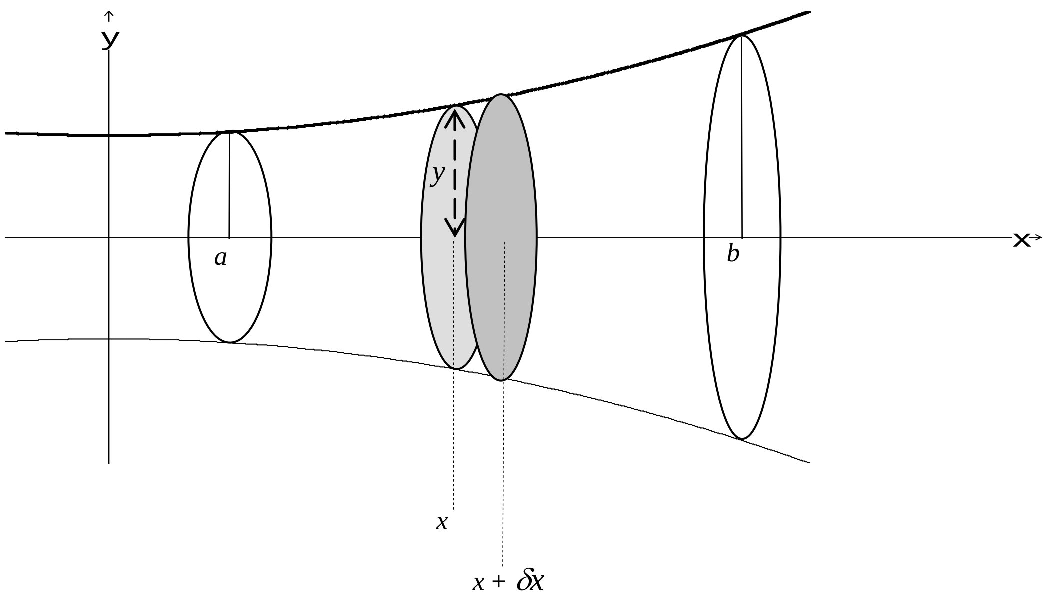
then the area of the rectangles is approximately the area under the curve

$$\sum_a^b y\,\delta x \cong \int_a^b y\,dx$$

and as $\delta x \to 0$ we have $\displaystyle\sum_a^b y\,\delta x \to \int_a^b y\,dx$

This last result is true for any integrable function y.

Volume of revolution



If the curve of $y = f(x)$ is rotated about the x–axis then the volume of the shape formed can be found by considering many slices each of width δx: one slice is shown.

The volume of this slice (a disc) is approximately $\pi y^2 \delta x$

$\Rightarrow$ Sum of volumes of all slices from a to b $\approx \sum_a^b \pi y^2 \, \delta x$

and as $\delta x \to 0$ we have (using the result above $\sum_a^b y \, \delta x \to \int_a^b y \, dx$)

$$\Rightarrow \text{ Volume } \approx \sum_a^b \pi y^2 \, \delta x \to \int_a^b \pi y^2 \, dx \quad .$$

Volume of revolution about the *x*–axis

Volume when $y = f(x)$, between $x = a$ and $x = b$, is rotated about the x–axis

is $V = \int_a^b \pi y^2 \, dx$.

Volume of revolution about the *y*–axis

Volume when $y = f(x)$, between $y = c$ and $y = d$, is rotated about the y–axis

is $V = \int_c^d \pi x^2 \, dy$.

Volume of rotation about the *y*-axis is not in the syllabus but is included for completeness.

Parametric integration

When x and y are given in parametric form we can find integrals using the techniques in integration by substitution.

$$\int y \; dx = \int y \frac{dx}{dt} \; dt$$ think of 'cancelling' the 'dt's

See the appendix for a justification of this result.

Example: If $x = \tan t$ and $y = \sin t$, find the area under the curve from $x = 0$ to $x = 1$.

Solution: The area $= \int y \; dx$ for some limits on x $= \int y \frac{dx}{dt} \; dt$ for limits on t.

We know that $y = \sin t$, and also that

$$x = \tan t \;\Rightarrow\; \frac{dx}{dt} = \sec^2 t$$

Finding limits for t: $x = 0 \Rightarrow t = 0$, and $x = 1 \Rightarrow t = \frac{\pi}{4}$

$$\Rightarrow \text{area} = \int_0^1 y \, dx = \int_0^{\frac{\pi}{4}} y \frac{dx}{dt} \, dt$$

$$= \int_0^{\frac{\pi}{4}} \sin t \sec^2 t \; dt = \int_0^{\frac{\pi}{4}} \tan t \; \sec t \; dt$$

$$= \Big[\sec t\Big]_0^{\frac{\pi}{4}} = \sqrt{2} - 1$$

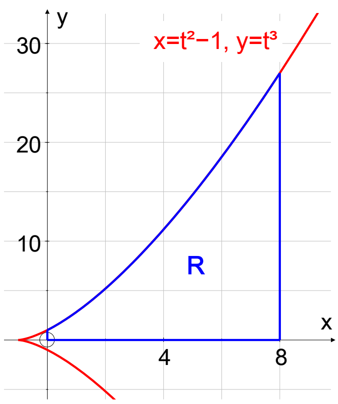
To find a volume of revolution we need $\int \pi y^2 \; dx$ and we proceed as above writing

$$\int \pi y^2 \; dx = \int \pi y^2 \frac{dx}{dt} \; dt$$

Example: The curve shown has parametric equations

$x = t^2 - 1, \quad y = t^3.$

The region, R, between $x = 0$ and $x = 8$ above the x-axis is rotated about the x-axis through 2π radians. Find the volume generated.



Solution: $V = \int_0^8 \pi y^2 \; dx$.

Change limits to t:

$x = 0 \Rightarrow t = \pm 1$ and $x = 8 \Rightarrow t = 3$,

but the curve is above the x-axis $\Rightarrow y = t^3 > 0 \Rightarrow t > 0, \Rightarrow t = +1$, or 3

also $y = t^3$, $x = t^2 - 1 \Rightarrow \frac{dx}{dt} = 2t$

$$\Rightarrow V = \int_0^8 \pi y^2 \, dx = \int_1^3 \pi y^2 \frac{dx}{dt} dt$$

$$= \int_1^3 \pi (t^3)^2 \times 2t \, dt = 2\pi \int_1^3 t^7 \, dt$$

$$= 2\pi \left[\frac{t^8}{8} \right]_1^3 = \frac{\pi}{4}\left(3^8 - 1\right)$$

Differential equations

Separating the variables

Example: Solve the differential equation $\frac{dy}{dx} = 3y + xy$.

Solution: $\frac{dy}{dx} = 3y + xy = y(3 + x)$

We first 'cheat' by separating the x s and y s onto different sides of the equation.

$\Rightarrow \quad \frac{1}{y} dy = (3 + x) \, dx$ and then put in the integral signs

$$\Rightarrow \quad \int \frac{1}{y} dy = \int 3 + x \, dx$$

$$\Rightarrow \quad \ln y = 3x + \tfrac{1}{2} x^2 + c.$$

See the appendix for a justification of this technique.

Exponential growth and decay

Example: A radio-active substance decays at a rate which is proportional to the mass of the substance present. Initially 25 grams are present and after 8 hours the mass has decreased to 20 grams. Find the mass after 1 day.

Solution: Let m grams be the mass of the substance at time t.

$\frac{dm}{dt}$ is the rate of **increase** of m so, since the mass is decreasing,

$\frac{dm}{dt} = -km \quad \Rightarrow \quad \frac{1}{m}\frac{dm}{dt} = -k$

$\Rightarrow \quad \int \frac{1}{m}\, dm = \int -k \, dt$

$\Rightarrow \quad \ln|m| = -kt + \ln|A|$ see ** below

$\Rightarrow \quad \ln\left|\frac{m}{A}\right| = -kt$

$\Rightarrow \quad m = Ae^{-kt}$.

When $t = 0$, $m = 25 \Rightarrow A = 25$

$\Rightarrow \quad m = 25e^{-kt}$.

When $t = 8$, $m = 20$

$\Rightarrow \quad 20 = 25e^{-8k} \quad \Rightarrow \quad e^{-8k} = 0{\cdot}8$

$\Rightarrow \quad -8k = \ln 0{\cdot}8 \quad \Rightarrow \quad k = 0{\cdot}027892943$

So when $t = 24$, $m = 25e^{-24 \times 0{\cdot}027892943} = 12{\cdot}8$.

Answer 12·8 grams after 1 day.

** Writing the arbitrary constant as $\ln|A|$ is a nice trick. If you don't like this you can write

$\ln|m| = -kt + c$

$\Rightarrow \quad |m| = e^{-kt+c} = e^c e^{-kt}$

$\Rightarrow \quad m = Ae^{-kt}$, writing $e^c = A$.

6 Vectors

Notation

The book and exam papers like writing vectors in the form

$\underline{a} = 3\underline{i} - 4\underline{j} + 7\underline{k}$.

It is allowed, and sensible, to re-write vectors in column form

i.e. $\underline{a} = 3\underline{i} - 4\underline{j} + 7\underline{k} = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}$.

Definitions, adding and subtracting, etcetera

A vector has both magnitude (length) and direction. If you always think of a vector as a translation you will not go far wrong.

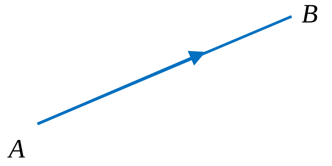
Directed line segments

The vector $\overrightarrow{AB}$ is the vector **from** A **to** B,

(or the translation which takes A to B).

This is sometimes called the

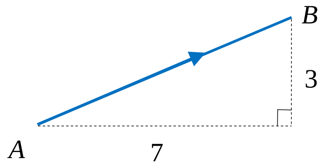
displacement vector from A to B.



Vectors in co-ordinate form

Vectors can also be thought of as column vectors,

thus in the diagram $\overrightarrow{AB} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.



Negative vectors

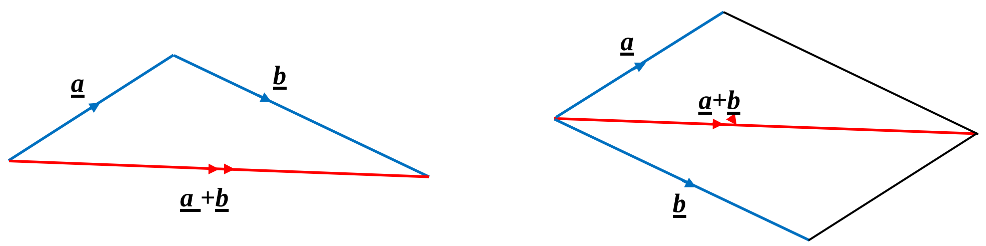
$\overrightarrow{AB}$ is the 'opposite' of $\overrightarrow{BA}$ and so $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{bmatrix} -7 \\ -3 \end{bmatrix}$.

Adding and subtracting vectors

(i) Using a diagram

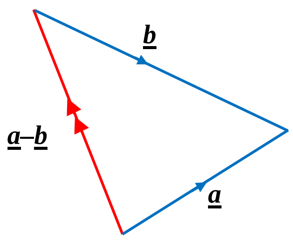
Geometrically this can be done using a triangle (or a parallelogram):

Adding:



The sum of two vectors is called the *resultant* of those vectors.

Subtracting:



(ii) Using coordinates

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}.$$

Parallel and non - parallel vectors

Parallel vectors

Two vectors are parallel if they have the same direction

$\Leftrightarrow$ one is a multiple of the other.

Example: Which two of the following vectors are parallel?

$$\begin{bmatrix} 6 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution: Notice that $\begin{bmatrix} 6 \\ -3 \end{bmatrix} = \frac{-3}{2} \times \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and so $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ is parallel to $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

but $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and so cannot be parallel to the other two vectors.

Example: Find a vector of length 15 in the direction of $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Solution: $\underline{\boldsymbol{a}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has length $a = |\underline{\boldsymbol{a}}| = \sqrt{4^2 + 3^2} = 5$

and so the required vector of length $15 = 3 \times 5$ is $3\underline{\boldsymbol{a}} = 3 \times \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ -9 \end{bmatrix}$.

Non-parallel vectors

If $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are not parallel and if $\alpha\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \gamma\underline{\boldsymbol{a}} + \delta\underline{\boldsymbol{b}}$, then

$\alpha\underline{\boldsymbol{a}} - \gamma\underline{\boldsymbol{a}} = \delta\underline{\boldsymbol{b}} - \beta\underline{\boldsymbol{b}} \Rightarrow (\alpha - \gamma)\underline{\boldsymbol{a}} = (\delta - \beta)\underline{\boldsymbol{b}}$

but $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are not parallel and one cannot be a multiple of the other

$\Rightarrow (\alpha - \gamma) = 0 = (\delta - \beta)$

$\Rightarrow \alpha = \gamma$ and $\delta = \beta$.

Example: If $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are not parallel and if

$\underline{\boldsymbol{b}} + 2\underline{\boldsymbol{a}} + \beta\underline{\boldsymbol{b}} = \alpha\underline{\boldsymbol{a}} + 3\underline{\boldsymbol{b}} - 5\underline{\boldsymbol{a}}$, find the values of α and β.

Solution: Since $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are not parallel, the coefficients of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ must 'balance out'

$\Rightarrow 2 = \alpha - 5 \Rightarrow \alpha = 7$ and $1 + \beta = 3 \Rightarrow \beta = 2$.

Modulus of a vector and unit vectors

Modulus

The *modulus* of a vector is its magnitude or length.

If $\overrightarrow{AB} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ then the modulus of $\overrightarrow{AB}$ is $AB = |\overrightarrow{AB}| = \sqrt{7^2 + 3^2} = \sqrt{58}$

Or, if $\underline{\boldsymbol{c}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ then the modulus of $\underline{\boldsymbol{c}}$ is $c = |\underline{\boldsymbol{c}}| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

Unit vectors

A *unit vector* is one with length 1.

Example: Find a unit vector in the direction of $\begin{bmatrix} -12 \\ 5 \end{bmatrix}$.

Solution: $\underline{\boldsymbol{a}} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$ has length $|\underline{\boldsymbol{a}}| = a = \sqrt{12^2 + 5^2} = 13$,

and so the required *unit* vector is $\frac{1}{13} \times \underline{\boldsymbol{a}} = \frac{1}{13} \times \begin{bmatrix} -12 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-12}{13} \\ \frac{5}{13} \end{bmatrix}$.

Position vectors

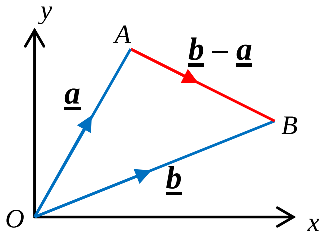
If A is the point $(-1, 4)$ then the position vector of A is the vector from the origin to A, usually written as $\overrightarrow{OA} = \underline{\boldsymbol{a}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

For two points A and B the position vectors are

$\overrightarrow{OA} = \underline{\boldsymbol{a}}$ and $\overrightarrow{OB} = \underline{\boldsymbol{b}}$

To find the vector $\overrightarrow{AB}$ go from $A \to O \to B$

giving $\overrightarrow{AB} = -\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$



Ratios

Example: A, B are the points $(2, 1)$ and $(4, 7)$. M lies on AB in the ratio $1 : 3$. Find the coordinates of M.

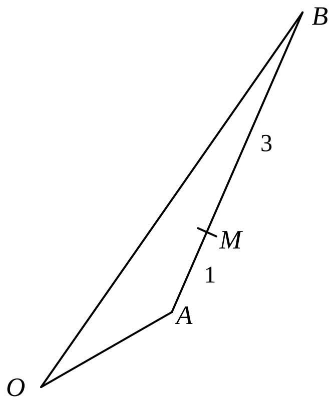
Solution : $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$\overrightarrow{AM} = \frac{1}{4}\overrightarrow{AB} = \frac{1}{4}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \cdot 5 \\ 1 \cdot 5 \end{bmatrix}$

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \cdot 5 \\ 1 \cdot 5 \end{bmatrix}$

$\Rightarrow \quad \overrightarrow{OM} = \begin{bmatrix} 2 \cdot 5 \\ 2 \cdot 5 \end{bmatrix}$

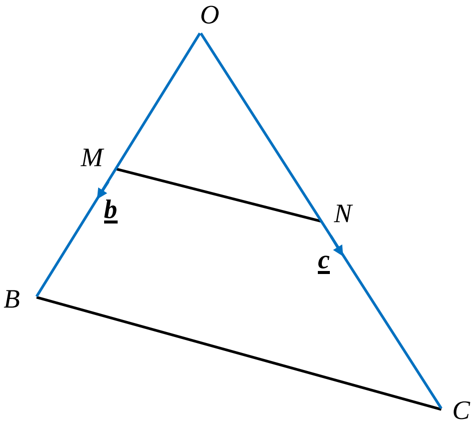
$\Rightarrow \quad M$ is $(2{\cdot}5, 2{\cdot}5)$



Proving geometrical theorems

Example: In a triangle OBC let M and N be the midpoints of OB and OC.

Prove that $BC = 2\,MN$ and that BC is parallel to MN.



Solution: Write the vectors $\overrightarrow{OB}$ as $\underline{\boldsymbol{b}}$, and $\overrightarrow{OC}$ as $\underline{\boldsymbol{c}}$.

Then $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\underline{\boldsymbol{b}}$

and $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\underline{\boldsymbol{c}}$.

To find $\overrightarrow{MN}$, go from M to O using $-\frac{1}{2}\underline{\boldsymbol{b}}$ and then from O to N using $\frac{1}{2}\underline{\boldsymbol{c}}$

$\Rightarrow \quad \overrightarrow{MN} = -\frac{1}{2}\underline{\boldsymbol{b}} + \frac{1}{2}\underline{\boldsymbol{c}}$

$\Rightarrow \quad \overrightarrow{MN} = \frac{1}{2}\underline{\boldsymbol{c}} - \frac{1}{2}\underline{\boldsymbol{b}}$

Also, to find $\overrightarrow{BC}$, go from B to O using $-\underline{\boldsymbol{b}}$ and then from O to C using $\underline{\boldsymbol{c}}$

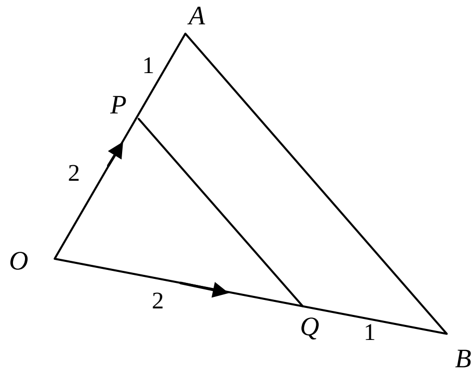
$\Rightarrow \overrightarrow{BC} = -\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}} = \underline{\boldsymbol{c}} - \underline{\boldsymbol{b}}$.

But $\overrightarrow{MN} = -\frac{1}{2}\underline{\boldsymbol{b}} + \frac{1}{2}\underline{\boldsymbol{c}} = \frac{1}{2}(\underline{\boldsymbol{c}} - \underline{\boldsymbol{b}}) = \frac{1}{2}\overrightarrow{BC}$

$\Rightarrow$ BC is parallel to MN

and BC is twice as long as MN.

Example: P lies on OA in the ratio 2 : 1, and Q lies on OB in the ratio 2 : 1. Prove that PQ is parallel to AB and that $PQ = {}^2/_3\,AB$.



Solution: Let $\underline{\boldsymbol{a}} = \overrightarrow{OA}$, and $\underline{\boldsymbol{b}} = \overrightarrow{OB}$

$\Rightarrow \quad \overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$

$\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA} = \frac{2}{3}\underline{\boldsymbol{a}}, \ \overrightarrow{OQ} = \frac{2}{3}\overrightarrow{OB} = \frac{2}{3}\underline{\boldsymbol{b}}$

and $\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = {}^2/_3\,\underline{\boldsymbol{b}} - {}^2/_3\,\underline{\boldsymbol{a}}$

$= {}^2/_3\,(\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$

$= {}^2/_3\,\overrightarrow{AB}$

$\Rightarrow \quad PQ$ is parallel to AB and

$\Rightarrow \quad PQ = {}^2/_3\,AB$.

Three dimensional vectors

Length, modulus or magnitude of a vector

The length, modulus or magnitude of the vector $\overrightarrow{OA} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is

$$\left|\overrightarrow{OA}\right| = |\underline{\boldsymbol{a}}| = a = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}},$$

a sort of three dimensional Pythagoras.

Distance between two points

To find the distance between A, (a_1, a_2, a_3) and B, (b_1, b_2, b_3) we need to find the length of the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix}$$

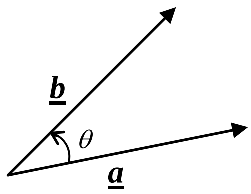
$$\Rightarrow \left|\overrightarrow{AB}\right| = AB = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

Scalar product

$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = ab \cos \theta$

where a and b are the lengths of $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$

and θ is the angle measured from $\underline{\boldsymbol{a}}$ to $\underline{\boldsymbol{b}}$.



Note that (i) $\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{a}} = aa \cos 0^{\circ} = a^2$

(ii) $\underline{\boldsymbol{a}} \cdot (\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{c}}$.

(iii) $\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{a}}$ since $\cos\theta = \cos(-\theta)$

In co-ordinate form

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 = ab \cos \theta$$

or $$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = ab \cos \theta.$$

Perpendicular vectors

If $\underline{a}$ and $\underline{b}$ are perpendicular then $\theta = 90^{\circ}$ and $\cos\theta = 0$

thus $\underline{a}$ perpendicular to $\underline{b}$ $\Rightarrow$ $\underline{a} \cdot \underline{b} = 0$

and $\underline{a} \cdot \underline{b} = 0$ $\Rightarrow$ *either* $\underline{a}$ is perpendicular to $\underline{b}$ *or* $\underline{a}$ or $\underline{b} = \underline{\mathbf{0}}$.

Example: Find the values of λ so that $\underline{a} = 3\underline{i} - 2\lambda\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} + \lambda\underline{j} + 6\underline{k}$ are perpendicular.

Solution: Since $\underline{a}$ and $\underline{b}$ are perpendicular $\underline{a} \bullet \underline{b} = 0$

$$\Rightarrow \begin{bmatrix} 3 \\ -2\lambda \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ \lambda \\ 6 \end{bmatrix} = 0 \quad \Rightarrow \quad 6 - 2\lambda^2 + 12 = 0$$

$$\Rightarrow \lambda^2 = 9 \quad \Rightarrow \quad \lambda = \pm 3.$$

Example: Find a vector which is perpendicular to $\underline{a}$, $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, and $\underline{b}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Solution: Let the vector $\underline{c}$, $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, be perpendicular to both $\underline{a}$ and $\underline{b}$.

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 0$$

$\Rightarrow$ $p - q + 2r = 0$ and $3p + q + r = 0.$

Adding these equations gives $4p + 3r = 0.$

Notice that there will never be a unique solution to these problems, so having eliminated one variable, q, we find p in terms of r, and then find q in terms of r.

$$\Rightarrow \quad p = \frac{-3r}{4} \quad \Rightarrow \quad q = \frac{5r}{4}$$

$\Rightarrow$ $\underline{c}$ is any vector of the form $\begin{bmatrix} \frac{-3r}{4} \\ \frac{5r}{4} \\ r \end{bmatrix}$,

and we choose a sensible value of $r = 4$ to give $\begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix}$.

Angle between vectors

Example: Find the angle between the vectors

$\overrightarrow{OA} = 4\underline{\boldsymbol{i}} - 5\underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}$ and $\overrightarrow{OA} = -\underline{\boldsymbol{i}} + 2\underline{\boldsymbol{j}} - 3\underline{\boldsymbol{k}}$, to the nearest degree.

Solution: First re-write as column vectors (if you want)

$$\underline{\boldsymbol{a}} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix} \text{ and } \underline{\boldsymbol{b}} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

$$a = |\underline{\boldsymbol{a}}| = \sqrt{4^2 + 5^2 + 2^2} = \sqrt{45} = 3\sqrt{5}, \quad b = |\underline{\boldsymbol{b}}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\text{and } \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = -4 - 10 - 6 = -20$$

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = ab\cos\theta \quad \Rightarrow \quad -20 = 3\sqrt{5} \times \sqrt{14}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{-20}{3\sqrt{70}} = -0.796819$$

$$\Rightarrow \quad \theta = 143^{\circ} \text{ to the nearest degree.}$$

Angle in a triangle

You must take care to find the angle requested, not '180 minus the angle requested'.

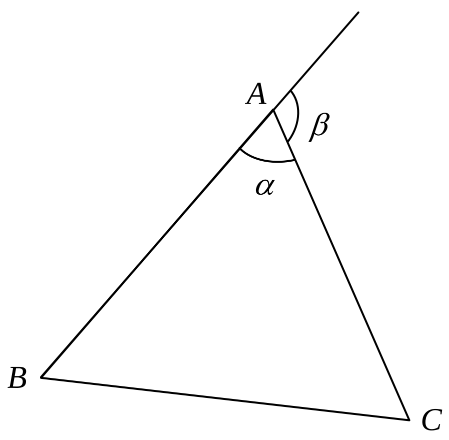
Example: A, (–1, 2, 4), B, (2, 3, 0), and C, (0, 2, -3) form a triangle. Find the angle $\angle BAC$.

Solution:

$\angle BAC = \alpha$, which is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

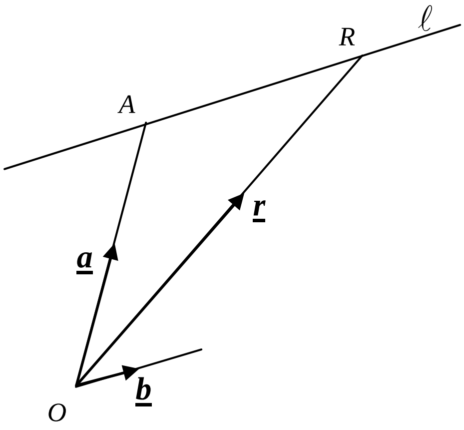
Note that the angle between $\overrightarrow{BA}$ and $\overrightarrow{AC}$ is the angle β, which is **not** the angle requested.

Then proceed as in the example above.



Vector equation of a straight line

$\underline{\boldsymbol{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is usually used as the position vector of a general point, R.



In the diagram the line ℓ passes through the point A and is parallel to the vector $\underline{\boldsymbol{b}}$.

To go from O to R first go to A, using $\underline{\boldsymbol{a}}$, and then from A to R using some multiple of $\underline{\boldsymbol{b}}$.

$\Rightarrow$ The equation of a straight line through the point A and parallel to the vector $\underline{\boldsymbol{b}}$ is

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}}.$$

Example: Find the vector equation of the line through the points M, (2, –1, 4), and N, (–5, 3, 7).

Solution: We are looking for the line through M (or N) which is parallel to the vector $\overrightarrow{MN}$.

$$\overrightarrow{MN} = \underline{\boldsymbol{n}} - \underline{\boldsymbol{m}} = \begin{bmatrix} -5 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$$

$\Rightarrow$ equation is $\underline{\boldsymbol{r}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$.

Example: Show that the point P, (–1, 7, 10), lies on the line

$$\underline{\boldsymbol{r}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The x co-ord of P is –1 and of the line is $1 - \lambda$

$\Rightarrow$ $-1 = 1 - \lambda \Rightarrow \lambda = 2$.

In the equation of the line this gives $y = -1 + 2 \times 4 = 7$ and $z = 4 + 2 \times 3 = 10$

$\Rightarrow$ P, (–1, 7, 10) does lie on the line.

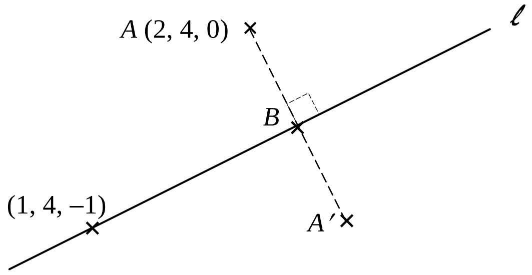
Geometrical problems

First DRAW a large diagram to see what is happening; this should then tell you how to use your vectors to solve the problem.

Example: Find A' the reflection of the point A (2, 4, 0) in the line ℓ, $\underline{\mathbf{r}} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

Solution: From the diagram we can see that AA' is perpendicular to ℓ.

So, if we can find the point B, where AB is perpendicular to ℓ, we will be able to find A', since $\overrightarrow{AB} = \overrightarrow{BA'}$.



B is a point on ℓ

$\Rightarrow \quad \overrightarrow{OB} = \underline{\mathbf{b}} = \begin{pmatrix} 1-\lambda \\ 4+2\lambda \\ -1+\lambda \end{pmatrix}$ for some value of λ.

$\underline{\mathbf{b}}$ is perpendicular to ℓ, and ℓ is parallel to $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow \quad \underline{\mathbf{b}} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} 1-\lambda \\ 4+2\lambda \\ -1+\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \quad -1 + \lambda + 8 + 4\lambda - 1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1$

$\Rightarrow \quad \underline{\mathbf{b}} = \begin{pmatrix} 1 - -1 \\ 4-2 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{AB} = \underline{\mathbf{b}} - \underline{\mathbf{a}} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$

$\Rightarrow \quad \overrightarrow{OA'} = \overrightarrow{OA} + 2\,\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\Rightarrow$ the reflection of A (2, 4, 0) in ℓ is A' (2, 0, –4).

Intersection of two lines

2 Dimensions

Example: Find the intersection of the lines

$\ell_1, \ \underline{\mathbf{r}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\ell_2, \ \underline{\mathbf{r}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: We are looking for values of λ and μ which give the same x and y co-ordinates on each line.

Equating x co-ords $\Rightarrow$ $2 - \lambda = 1 + \mu$

equating y co-ords $\Rightarrow$ $\underline{3 + 2\lambda = 3 - \mu}$

Adding $\Rightarrow$ $5 + \lambda = 4$ $\Rightarrow$ $\lambda = -1$ $\Rightarrow$ $\mu = 2$

$\Rightarrow$ lines intersect at (3, 1).

3 Dimensions

This is similar to the method for 2 dimensions with one important difference – you can **not** be certain whether the lines intersect without checking.

You will always (or nearly always) be able to find values of λ and μ by equating x coordinates and y coordinates but the ***z* coordinates might or might not be equal and must be checked.**

Example: Investigate whether the lines

$\ell_1, \quad \underline{\mathbf{r}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\ell_2, \ \underline{\mathbf{r}} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ intersect

and if they do find their point of intersection.

Solution: **If** the lines intersect we can find values of λ and μ to give the same x, y and z coordinates in each equation.

Equating x coords $\Rightarrow$ $2 - \lambda = -3 + \mu$, **I**

equating y coords $\Rightarrow$ $1 + 2\lambda = 1 + 3\mu$, **II**

equating z coords $\Rightarrow$ $3 + \lambda = 5 + \mu$. **III**

$2 \times$ **I** + **II** $\Rightarrow$ $5 = -5 + 5\mu$ $\Rightarrow$ $\mu = 2$, in **I** $\Rightarrow$ $\lambda = 3$.

We must now check to see if we get the same point for the values of λ and μ

In ℓ_1, $\lambda = 3$ gives the point (–1, 7, 6);

in ℓ_2, $\mu = 2$ gives the point (–1, 7, 7).

The x and y co-ords are equal (as expected!), but the z co-ordinates are different and so the lines do **not** intersect.

7 Appendix

Binomial series $(1 + x)^n$ for any n – proof

Suppose that

$$f(x) = (1+x)^n = a + bx + cx^2 + dx^3 + ex^4 + \ldots$$

put $x = 0$, $\Rightarrow \quad 1 = a$

$\Rightarrow \quad f'(x) = n(1+x)^{n-1} = b + 2cx + 3dx^2 + 4ex^3 + \ldots$

put $x = 0$, $\Rightarrow \quad n = b$

$\Rightarrow \quad f''(x) = n(n-1)(1+x)^{n-2} = 2c + 3 \times 2dx + 4 \times 3ex^2 + \ldots$

put $x = 0$, $\Rightarrow \; n(n-1) = 2c \;\Rightarrow\; \frac{n(n-1)}{2!} = c$

$\Rightarrow \quad f'''(x) = n(n-1)(n-2)(1+x)^{n-3} = 3 \times 2d + 4 \times 3 \times 2ex + \ldots$

put $x = 0$, $\Rightarrow \; n(n-1)(n-2) = 3 \times 2d \;\Rightarrow\; \frac{n(n-1)(n-2)}{3!} = d$

Continuing this process, we have

$\frac{n(n-1)(n-2)(n-3)}{4!} = e$ and $\frac{n(n-1)(n-2)(n-3)(n-4)}{5!} = f$, etc.

giving $f(x) = (1+x)^n$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}x^5 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

Showing that this is convergent for $|x| < 1$, is more difficult!

Derivative of x^q for q rational

Suppose that q is any rational number, $q = \frac{r}{s}$, where r and s are integers, $s \neq 0$.

Then $\quad y = x^q = x^{r/s} \Rightarrow y^s = x^r$

Differentiating with respect to $x \;\Rightarrow\; s \times y^{s-1}\frac{dy}{dx} = r \times x^{r-1}$

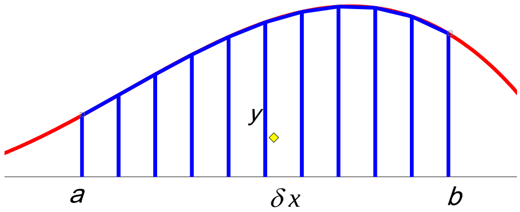
$\Rightarrow \quad \frac{dy}{dx} = \frac{r}{s} \times \frac{x^{r-1}}{y^{s-1}} = \frac{r}{s} \times \frac{x^{r-1}}{y^s} \times y = q \times \frac{x^{r-1}}{y^s} \times y \qquad$ since $q = \frac{r}{s}$

$\Rightarrow \quad \frac{dy}{dx} = q \times \frac{x^{r-1}}{x^r} \times x^q = qx^{q-1} \qquad$ since $y^s = x^r$ and $y = x^q$

which follows the rule found for x^n, where n is an integer.

$\int \frac{1}{x}\, dx$ for negative limits

We know that the 'area' under any curve, from $x = a$ to $x = b$ is approximately

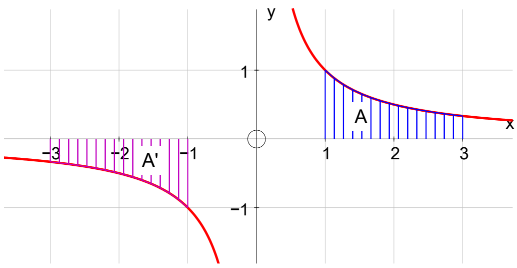


$$\sum_a^b y\ \delta x \quad \to \quad \int_a^b y\ dx, \qquad as\ \delta x \to 0$$

If the curve is above the x-axis, all the y values are positive, and if $a < b$ then all values of δx are positive, and so the integral is positive.

$$\int_{-a}^{-b} \frac{1}{x}\ dx \ = \ \Big[\ln|x|\Big]_{-a}^{-b} \ = \ \ln b - \ln a$$

Example: Find $\displaystyle\int_{-1}^{-3} \frac{1}{x}\ dx$.



Solution: The integral wanted is shown as A' in the diagram.

By symmetry $|A'| = A$ (A positive)

and we need to decide whether the integral is $+A$ or $-A$.

From $x = -1$ to $x = -3$, we are going in the direction of x **decreasing**

$\Rightarrow$ all δx are negative.

And the graph is below the x-axis,

$\Rightarrow$ the y values are negative, $\Rightarrow$ $y\ \delta x$ is **positive**

$\Rightarrow$ $\sum_{-1}^{-3} y\ \delta x > 0 \ = \ \int_{-1}^{-3} y\ dx > 0$

$\Rightarrow$ the integral is **positive** and equal to A.

The integral, $A' = A = \displaystyle\int_1^3 \frac{1}{x}\ dx = \big[\ln x\big]_1^3 = \ln 3 - \ln 1 = \ln 3$

$$\Rightarrow \quad A' = \int_{-1}^{-3} \frac{1}{x}\ dx = \ln 3$$

Notice that this is what we get if we write $\ln|x|$ in place of $\ln x$

$$\int_{-1}^{-3} \frac{1}{x}\ dx = \big[\ln|x|\big]_{-1}^{-3} = \ln 3 - \ln 1 = \ln 3$$

As it will always be possible to use symmetry in this way, since we can never have one positive and one negative limit (because there is a discontinuity at $x = 0$), it is correct to write $\ln|x|$ for the integral of $^1/_x$.

Integration by substitution – why it works

We show the general method with an example.

$$y = \int x^2\sqrt{1+x^3}\;\; dx$$

$\Rightarrow \quad \dfrac{dy}{dx} = x^2\sqrt{1+x^3}$ integrand $= x^2\sqrt{1+x^3}$

Choose $u = 1 + x^3$

$\Rightarrow \quad \dfrac{du}{dx} = 3x^2 \quad \Longrightarrow \quad \dfrac{dx}{du} = \dfrac{1}{3x^2}$ rearrange to give $dx = \dfrac{du}{3x^2}$

But $\quad \dfrac{dy}{du} = \dfrac{dy}{dx} \times \dfrac{dx}{du}$

$\Rightarrow \quad y = \displaystyle\int \dfrac{dy}{dx} \times \dfrac{dx}{du}\;\; du$

$\dfrac{dy}{dx} = x^2\sqrt{u}$ leave the x^2 because it appears in dx

$y = \displaystyle\int x^2\sqrt{u} \times \dfrac{1}{3x^2}\;\; du$ this is the same as writing the integrand in terms of u, and then replacing dx by $\frac{dx}{du}\, du = \frac{du}{3x^2}$

The essential part of this method, writing the integrand in terms of u, and then replacing dx by $\frac{dx}{du}\, du$, will be the same for all integrations by substitution.

Parametric integration

This is similar to integration by substitution.

$A = \displaystyle\int y\;dx \quad \Rightarrow \quad \dfrac{dA}{dx} = y$

$\Rightarrow \quad \dfrac{dA}{dt} \times \dfrac{dt}{dx} = y$

$\Rightarrow \quad \dfrac{dA}{dt} = y \times \dfrac{dx}{dt}$

$\Rightarrow \quad A = \displaystyle\int y\,\dfrac{dx}{dt}\;\; dt$

Separating the variables – why it works

We show this with an example.

If $y = 6y^3$ then $\frac{dy}{dx} = 18y^2 \frac{dy}{dx}$

and so $\int 18y^2 \frac{dy}{dx}\ dx = \int 18y^2\ dy = 6y^3 + c$

Notice that we 'cancel' the dx.

Example: Solve $\frac{dy}{dx} = x^2 \sec y$

Solution: $\frac{dy}{dx} = x^2 \sec y$

$\Rightarrow \quad \cos y \frac{dy}{dx} = x^2$

$\Rightarrow \quad \int \cos y \frac{dy}{dx}\ dx = \int x^2\ dx$

$\Rightarrow \quad \int \cos y\ dy = \int x^2\ dx$ 'cancelling' the dx

$\Rightarrow \quad \sin y = \frac{1}{3}x^3 + c$

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