# Pure Core 1

# Revision Notes

June 2014

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# 1 Algebra

## Indices

### Rules of indices

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$a^0 = 1$

$a^{-n} = \dfrac{1}{a^n}$

$a^{1/n} = \sqrt[n]{a}$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

*Examples:*

(i) $5^{-3} \times 5^4 = 5^{-3+4} = 5^1 = 5.$

$7^{-4} \times 7^{-2} = 7^{-4+-2} = 7^{-4-2} = 7^{-6} = \dfrac{1}{7^6}.$

(ii) $3^5 \div 3^{-2} = 3^{5--2} = 3^{5+2} = 3^7.$

$9^{-4} \div 9^6 = 9^{-4-6} = 9^{-10} = \dfrac{1}{9^{10}}$

$11^{-3} \div 11^{-5} = 11^{-3--5} = 11^{-3+5} = 11^2 = 121$

(iii) $(6^{-3})^4 = 6^{-3\times 4} = 6^{-12} = \dfrac{1}{6^{12}}.$

(iv) $64^{2/3} = \left(64^{1/3}\right)^2 = (4)^2 = 16$

(v) $125^{-2/3} = \dfrac{1}{125^{2/3}}$ since *minus* means *turn upside down*

$= \dfrac{1}{5^2},$ since 3 on bottom of fraction is cube root, $\sqrt[3]{125} = 5$

$= \dfrac{1}{25}$

*Example:* Express $(16^a) \div (8^b)$ as power of 2.

*Solution:* $(16^a) \div (8^b) = (2^4)^a \div (2^3)^b = 2^{4a} \div 2^{3b} = 2^{4a-3b}.$

*Example:* Find $x$ if $9^{2x} = 27^{x+1}.$

*Solution:* First notice that $9 = 3^2$ and $27 = 3^3$ and so

$9^{2x} = 27^{x+1} \quad \Rightarrow \quad (3^2)^{2x} = (3^3)^{x+1}$

$\Rightarrow \quad 3^{4x} = 3^{3x+3}$

$\Rightarrow \quad 4x = 3x + 3 \quad \Rightarrow \quad x = 3.$

## *Surds*

A surd is a 'nasty' root – i.e. a root which is not rational.

Thus $\sqrt{64} = 8$, $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, $\sqrt[5]{-243} = -3$ are rational and not surds

and $\sqrt{5}$, $\sqrt[5]{45}$, $\sqrt[3]{-72}$ are irrational and **are surds**.

### Simplifying surds

*Example:* To simplify $\sqrt{50}$ we notice that $50 = 25 \times 2 = 5^2 \times 2$

$\Rightarrow \quad \sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$.

*Example:* To simplify $\sqrt[3]{40}$ we notice that $40 = 8 \times 5 = 2^3 \times 5$

$\Rightarrow \quad \sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2 \times \sqrt[3]{5}$.

### Rationalising the denominator

*Rationalising* means getting rid of surds.

We remember that multiplying $(a + b)$ by $(a - b)$ gives $a^2 - b^2$ which has the effect of squaring **both** $a$ and $b$ at the same time!!

*Example:* Rationalise the denominator of $\frac{2+3\sqrt{5}}{3-\sqrt{5}}$ .

*Solution:* $\frac{2+3\sqrt{5}}{3-\sqrt{5}} = \frac{2+3\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$

$= \frac{6+3\sqrt{5}\sqrt{5}+9\sqrt{5}+2\sqrt{5}}{3^2-(\sqrt{5})^2}$

$= \frac{21+11\sqrt{5}}{4}$ .

# 2 Quadratic functions

A quadratic function is a function $ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and the highest power of $x$ is 2.

The numbers $a$ and $b$ are called the *coefficients* of $x^2$ and $x$, and $c$ is the *constant term*.

**Completing the square.**

*a*) The coefficient of $x^2$ must be +1.

*b*) Halve the coefficient of $x$, square it then add it and subtract it.

*Example:* Complete the square in $x^2 - 6x + 7$.

*Solution:* *a*) The coefficient of $x^2$ is already +1,

*b*) the coefficient of $x$ is $-6$, halve it to give $-3$ then square to give 9 which is then added and subtracted

$$x^2 - 6x + 7 = x^2 - 6x + (-3)^2 - 9 + 7$$
$$= (x - 3)^2 - 2.$$

*Notice* the *min*imum value of $x^2 - 6x + 7$ is $-2$ when $x = 3$,
since the *min*imum value of $(x - 3)^2$ is 0

$\Rightarrow$ the *vertex* of the graph of $y = x^2 - 6x + 7$ (parabola) is at $(3, -2)$.

*Example:* Complete the square in $-3x^2 - 24x + 5$.

*Solution:* *a*) The coefficient of $x^2$ is not +1, so we must take out a factor of $-3$ first, and then go on to step *b*).

*a*) $-3x^2 - 24x + 5 = -3(x^2 + 8x) + 5$

*b*) $= -3(x^2 + 8x + 4^2 - 16) + 5$

$= -3(x + 4)^2 + 48 + 5$

$= -3(x + 4)^2 + 53$

*Notice* the *max*imum value of $-3x^2 - 24x + 5$ is $+53$ when $x = -4$,
since the *max*imum value of $(x + 4)^2$ is 0

$\Rightarrow$ the *vertex* of the graph of $y = -3x^2 - 24x + 5$ (parabola) is at $(-4, 53)$.

**Factorising quadratics**

If all the coefficients are integers (whole numbers) then the factors will also have integer coefficients. There is no point in looking for factors with fractions as coefficients.

*Example:* Factorise $10x^2 + 11x - 6$.

*Solution:* Looking at the $10x^2$ and the $-6$ we see that possible factors are

| | | | |
|---|---|---|---|
| $(10x \pm 1)$, | $(10x \pm 2)$, | $(10x \pm 3)$, | $(10x \pm 6)$, |
| $(5x \pm 1)$, | $(5x \pm 2)$, | $(5x \pm 3)$, | $(5x \pm 6)$, |
| $(2x \pm 1)$, | $(2x \pm 2)$, | $(2x \pm 3)$, | $(2x \pm 6)$, |
| $(x \pm 1)$, | $(x \pm 2)$, | $(x \pm 3)$, | $(x \pm 6)$, |

Also the $-6$ tells us that the factors must have opposite signs, and by trial and error or common sense

$$10x^2 + 11x - 6 = (2x + 3)(5x - 2).$$

## *Solving quadratic equations.*

**By factorising.**

*Example:* $x^2 - 5x + 6 = 0 \Rightarrow (x - 3)(x - 2) = 0$

$\Rightarrow$ $x - 3 = 0$ or $x - 2 = 0$ $\Rightarrow$ $x = 3$ or $x = 2$.

*Example:* $x^2 + 8x = 0 \Rightarrow x(x + 8) = 0$

$\Rightarrow$ $x = 0$ or $x + 8 = 0$

$\Rightarrow$ $x = 0$ or $x = -8$

**N.B.** Do **not** divide through by $x$ first: you will lose the root of $x = 0$.

**By completing the square**

*Example:* $x^2 - 6x - 4 = 0$

$\Rightarrow$ $x^2 - 6x = 4$ coefficient of $x^2$ is +1, halve $-6$, square and add 9 to both sides

$\Rightarrow$ $x^2 - 6x + 9 = 4 + 9$

$\Rightarrow$ $(x - 3)^2 = 13$

$(x - 3) = \pm\sqrt{13} \Rightarrow x = 3 \pm \sqrt{13} = -0{\cdot}61$ or $6{\cdot}61$ to 3 S.F.

**By using the formula**

For a proof of the formula, see the appendix

Always try to factorise first.

$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

*Example:* Solve the equation $3x^2 - x - 5 = 0$

*Solution:* $3x^2 - x - 5$ will not factorise,

so we use the formula with $a = 3$, $b = -1$, $c = -5$

$\Rightarrow \quad x = \frac{-(-1) \pm \sqrt{(-1)^2-4\times3\times(-5)c}}{2\times3} = -1{\cdot}135$ or $+1{\cdot}468$ to 3 D.P.

**The discriminant, $b^2 - 4ac$**

The quadratic equation

$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ will have

i) two distinct real roots if $b^2 - 4ac > 0$

ii) only one real root if $b^2 - 4ac = 0$ (or two coincident real roots)

iii) no real roots if $b^2 - 4ac < 0$

*Example:* For what values of $k$ does the equation $3x^2 - kx + 5 = 0$ have

i) two distinct real roots,

ii) exactly one real root

iii) no real roots.

*Solution:* The discriminant $b^2 - 4ac = (-k)^2 - 4 \times 3 \times 5 = k^2 - 60$

$\Rightarrow$ i) for two distinct real roots $k^2 - 60 > 0$

$\Rightarrow \quad k^2 > 60 \quad \Rightarrow \quad k < -\sqrt{60}$, or $k > +\sqrt{60}$

and ii) for only one real root $k^2 - 60 = 0$

$\Rightarrow \quad k = \pm\sqrt{60}$

and iii) for no real roots $k^2 - 60 < 0$

$\Rightarrow \quad -\sqrt{60} < k < +\sqrt{60}$.

**Miscellaneous quadratic equations**

*Example:* Solve $3^{2x} - 10 \times 3^x + 9 = 0$

*Solution:* Notice that $3^{2x} = (3^x)^2$

so put $y = 3^x$ to give

$y^2 - 10y + 9 = 0$

$\Rightarrow \quad (y - 9)(y - 1) = 0$

$\Rightarrow \quad y = 9$ or $y = 1$

$\Rightarrow \quad 3^x = 9$ or $3^x = 1$

$\Rightarrow \quad x = 2$ or $x = 0$.

*Example:* Solve $y - 3\sqrt{y} + 2 = 0$.

*Solution:* Put $\sqrt{y} = x$ to give

$x^2 - 3x + 2 = 0$

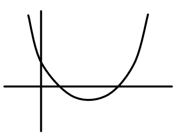
$\Rightarrow \quad (x - 2)(x - 1)$

$\Rightarrow \quad x = 2$ or $1$

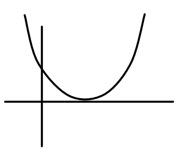
$\Rightarrow \quad y = x^2 = 4$ or $1$.

## *Quadratic graphs*

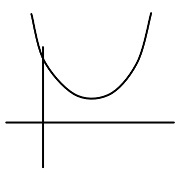
*a*) If $a > 0$ the parabola will be 'the right way up'



$b^2 - 4ac > 0$
2 distinct real roots

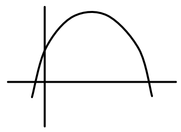


$b^2 - 4ac = 0$
only 1 real root

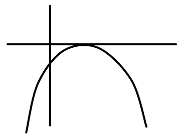


$b^2 - 4ac < 0$
no real roots

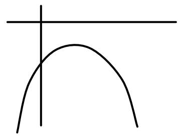
b) If $a < 0$ the parabola will be 'the wrong way up'



$b^2 - 4ac > 0$
2 distinct real roots



$b^2 - 4ac = 0$
only 1 real root



$b^2 - 4ac < 0$
no real roots

**Note:** When sketching the curve of a quadratic function you should always show the value on the *y*-axis,

if you have factorised you should show the values where it meets the *x*-axis,

if you have completed the square you should give the co-ordinates of the vertex.

## *Simultaneous equations*

**Two linear equations**

*Example:* Solve $3x - 2y = 4$ **I**

$4x + 7y = 15$ **II**

*Solution:* Make the coefficients of *x* (or *y*) equal then add or subtract the equations to eliminate *x* (or *y*).

$4 \times \mathbf{I} \Rightarrow 12x - 8y = 16$

$3 \times \mathbf{II} \Rightarrow \underline{12x + 21y = 45}$ Subtracting

$-29y = -29$

$\Rightarrow y = 1$

Put $y = 1$ in **I** $\Rightarrow 3x = 4 + 2y = 4 + 2 \Rightarrow x = 2$

Check in equation two: L.H.S. = $4 \times 2 + 7 \times 1 = 15$ = R.H.S.

*Answer* : $x = 2, \; y = 1$

## One linear equation and one quadratic.

Find $x$ (or $y$) from the linear equation and substitute in the quadratic equation.

*Example:* Solve $x - 2y = 3$ **I**

$x^2 - 2y^2 - 3y = 5$ **II**

*Solution:* From **I** $x = 2y + 3$

Substitute in **II**

$\Rightarrow \quad (2y + 3)^2 - 2y^2 - 3y = 5$

$\Rightarrow \quad 4y^2 + 12y + 9 - 2y^2 - 3y = 5$

$\Rightarrow \quad 2y^2 + 9y + 4 = 0 \quad \Rightarrow \quad (2y + 1)(y + 4) = 0$

$\Rightarrow \quad y = -½$ or $y = -4$

$\Rightarrow \quad x = 2$ or $x = -5$ using **I**

Check in the quadratic equation

When $x = 2$ and $y = -½$

L.H.S. $= 2^2 - 2(-½)^2 - 3(-½) = 5 =$ R.H.S.

and when $x = -5$ and $y = -4$

L.H.S. $= (-5)^2 - 2(-4)^2 - 3(-4) = 25 - 32 + 12 = 5 =$ R.H.S.

*Answer:* $x = 2,\ y = -½$ or $x = -5,\ y = -4$

## *Inequalities*

### Linear inequalities

Solving algebraic inequalities is just like solving equations, add, subtract, multiply or divide the same number to, from, etc. BOTH SIDES

EXCEPT - if you multiply or divide both sides by a NEGATIVE number then you must TURN THE INEQUALITY SIGN ROUND.

*Example:* Solve $3 + 2x < 8 + 4x$

*Solution:* sub 3 from B.S. $\Rightarrow \quad 2x < 5 + 4x$
sub $4x$ from B.S. $\Rightarrow \quad -2x < 5$
divide B.S. by $-2$ and **turn the inequality sign round**
$\Rightarrow \quad x > -2{\cdot}5$.

### Quadratic inequalities, $x^2 > k$ or $< k$

*Example:* Solve $x^2 > 16$

*Solution:* We must be careful here since the square of a negative number is positive giving the full range of solutions as

$\Rightarrow \qquad x < -4$ **or** $x > +4$.

### Quadratic inequalities, $ax^2 + bx + c > k$ or $< k$

Always sketch a graph and find where the curve meets the $x$-axis

*Example:* Find the values of $x$ which satisfy

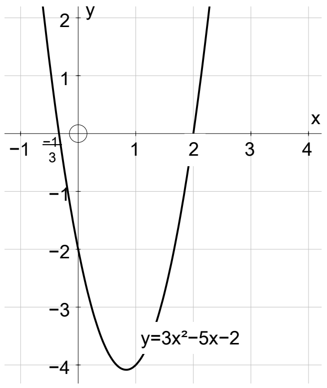
$3x^2 - 5x - 2 \geq 0$.

*Solution:*

solving $\qquad 3x^2 - 5x - 2 = 0$

$\Rightarrow \qquad (3x + 1)(x - 2) = 0$

$\Rightarrow \qquad x = -{}^1/_3$ or $2$




We want the part of the curve $y = 3x^2 - 5x - 2$ which is above or on the $x$-axis

$\Rightarrow \quad x \leq -{}^1/_3$ **or** $x \geq 2$

# 3 Coordinate geometry

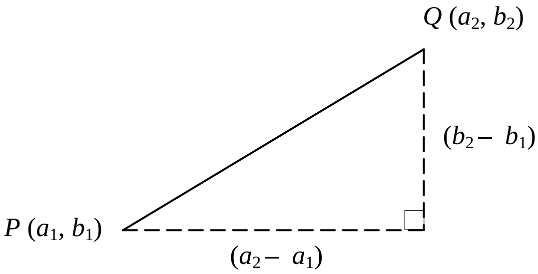
## *Distance between two points*

Distance between $P\ (a_1, b_1)$ and $Q\ (a_2, b_2)$ is $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

## *Gradient*

Gradient of $PQ$ is $m = \dfrac{b_2 - b_1}{a_2 - a_1}$

## *Equation of a line*




Equation of the line $PQ$, above, is $y = mx + c$
and use a point to find $c$

or the equation of the line with gradient $m$ through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

or the equation of the line through the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$ (you do not need to know this one!!)

## *Parallel and perpendicular lines*

Two lines are parallel if they have the same gradient
and they are perpendicular if the product of their gradients is –1.

*Example:* Find the equation of the line through $(4\frac{1}{2}, 1)$ and perpendicular to the line joining the points $A\ (3, 7)$ and $B\ (6, -5)$.

*Solution:* Gradient of $AB$ is $\frac{7--5}{3-6} = -4$

$\Rightarrow$ gradient of line perpendicular to $AB$ is $\frac{1}{4}$, product of perpendicular gradients is –1

so we want the line through $(4\frac{1}{2}, 1)$ with gradient $\frac{1}{4}$.

Using $y - y_1 = m(x - x_1)$ $\Rightarrow$ $y - 1 = \frac{1}{4}(x - 4\frac{1}{2})$

$\Rightarrow$ $4y - x = -\frac{1}{2}$ or $2x - 8y - 1 = 0$.

# 4 Sequences and series

A *sequence* is any list of numbers.

## *Definition by a formula $x_n = f(n)$*

*Example:* The definition $x_n = 3n^2 - 5$ gives

$x_1 = 3 \times \mathbf{1}^2 - 5 = -2$, $x_2 = 3 \times \mathbf{2}^2 - 5 = 7$, $x_3 = 3 \times \mathbf{3}^2 - 5 = 22$, … .

## *Definitions of the form $x_{n+1} = f(x_n)$*

These have two parts:–

(i) a starting value (or values)

(ii) a method of obtaining each term from the one(s) before.

*Examples:* (i) The definition $x_1 = 3$ and $x_n = 3x_{n-1} + 2$

defines the sequence 3, 11, 35, 107, . . .

(ii) The definition $x_1 = 1$, $x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$

defines the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 65, . . . .

This is the Fibonacci sequence.

## *Series and Σ notation*

A *series* is the sum of the first so many terms of a sequence.

For a sequence whose $n$th term is $x_n = 2n + 3$ the sum of the first $n$ terms is a series

$S_n = x_1 + x_2 + x_3 + x_4 \ldots + x_n = 5 + 7 + 9 + 11 + \ldots + (2n + 3)$

This is written in Σ notation as $S_n = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (2i + 3)$ and is a *finite* series of $n$ terms.

An *infinite* series has an infinite number of terms $S_\infty = \sum_{i=1}^{\infty} x_i$ .

## *Arithmetic series*

An *arithmetic series* is a series in which each term is a constant amount bigger (or smaller) than the previous term: this *constant amount* is called the *common difference*.

*Examples:* 3, 7, 11, 15, 19, 23, . . . . with common difference 4

28, 25, 22, 19, 16, 13, . . . with common difference –3.

*Generally* an arithmetic series can be written as

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + \ldots$ upto $n$ terms,

where the first term is $a$ and the common difference is $d$.

The $n$th term $x_n = a + (n - 1)d$

The *sum* of the first $n$ terms of the above arithmetic series is

$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$, or $S_n = \frac{n}{2}\left(a + l\right)$ where $l$ is the **last** term.

*Example:* Find the $n$th term and the sum of the first 100 terms of the arithmetic series with 3rd term 5 and 7th term 17.

*Solution:* $x_7 = x_3 + 4d$

$\Rightarrow \quad 17 = 5 + 4d \quad \Rightarrow \quad d = 3$

$\Rightarrow \quad x_1 = x_3 - 2d \quad \Rightarrow \quad x_1 = 5 - 6 = -1$

$\Rightarrow \quad n$th term $x_n = a + (n - 1)d = -1 + (n - 1) \times 3$

and $\Rightarrow \quad S_{100} = \frac{100}{2} \times (2 \times (-1) + (100 - 1) \times 3) = 14750.$

**Proof of the formula for the sum of an arithmetic series**

You **must** know this proof.

First write down the general series and then write it down in reverse order

$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d$

$\Rightarrow \quad S_n = (a + (n - 1)d + (a + (n - 2)d) + (a + (n - 3)d) + \ldots + a$

**ADD**

$\Rightarrow \quad 2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots (2a + (n - 1)d)$

$\Rightarrow \quad 2S_n = n(2a + (n - 1)d)$

$\Rightarrow \quad S_n = \frac{n}{2} \times (2a + (n - 1)d)$. **I**

This can be written as

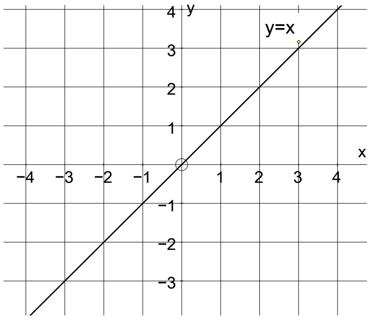
$S_n = \frac{n}{2} \times \{a + (a + (n - 1)d)\}$

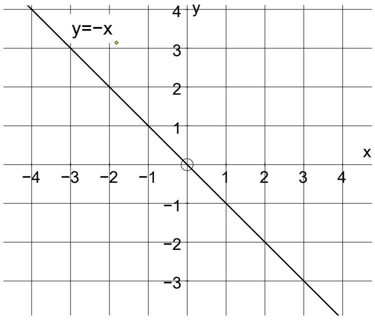
$\Rightarrow \quad S_n = \frac{n}{2} \times (a + l)$, where $l$ is the last term. **II**

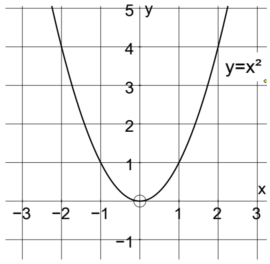
You should know both **I** and **II**.

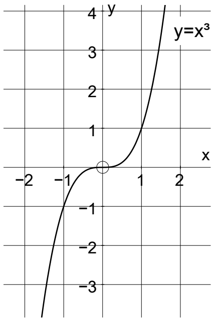
# 5 Curve sketching

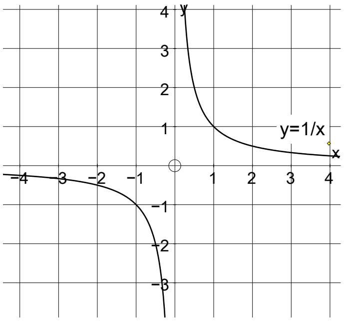
## *Standard graphs*

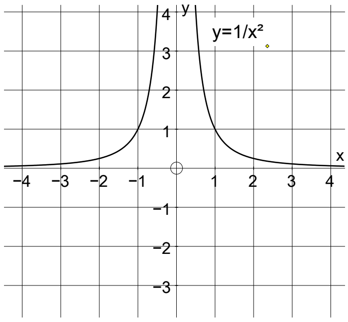


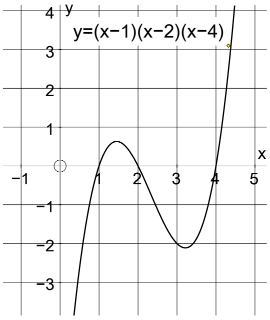



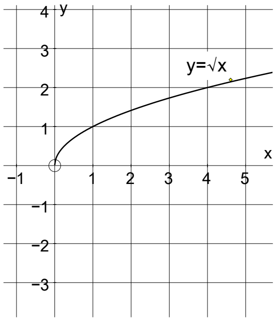



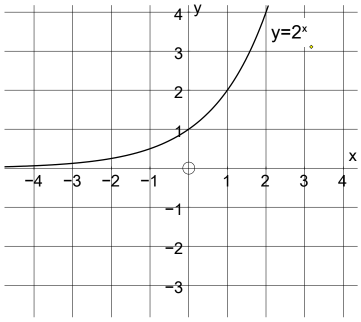



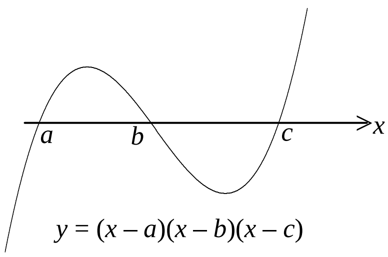

$y = 3x^2$ is like $y = x^2$ but steeper: similarly for $y = 5x^3$ and $y = {}^7/_x$, etc.

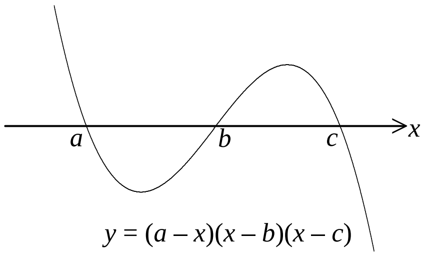
**Cubic graphs**

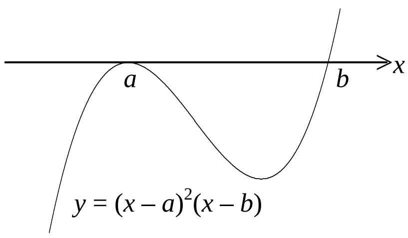
Some types of cubic graph are shown below:

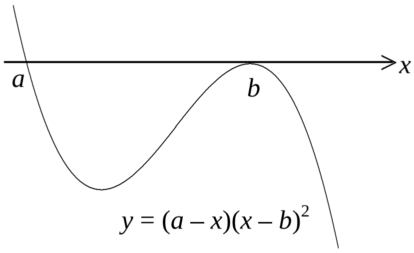
Coefficient of $x^3$ is +1

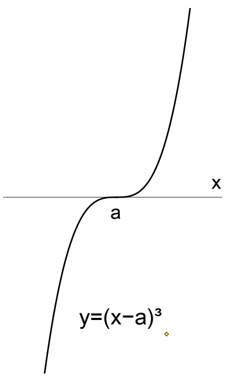
Coefficient of $x^3$ is –1

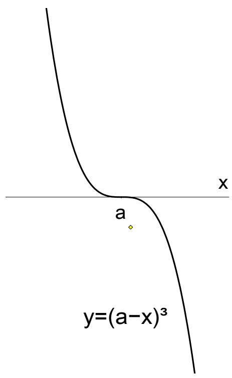


## *Transformations of graphs*

### Translations

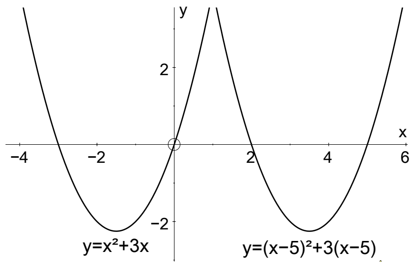
(i) If the graph of $y = x^2 + 3x$ is translated through $\binom{0}{5}$, +5 in the $y$-direction, the equation of the new graph is

$$y = x^2 + 3x + 5;$$

and, in general, the graph of $y = f(x)$ after a translation through $\binom{0}{b}$ has equation $y = f(x) + b$.

(ii) If the graph of $y = x^2 + 3x$ is translated through $\binom{5}{0}$, +5 in the $x$-direction, the equation of the new graph is

$$y = (x - 5)^2 + 3(x - 5).$$

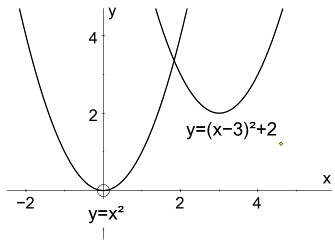



and, in general, the graph of $y = f(x)$ after a translation through $\binom{a}{0}$ has equation $y = f(x - a)$.

We replace $x$ by $(x - a)$ everywhere in the formula for $y$:
note the minus sign, $-a$, which seems wrong but is correct!

*Example:*

The graph of $y = (x - 3)^2 + 2$ is the graph of $y = x^2$ after a translation of $\binom{+3}{+2}$

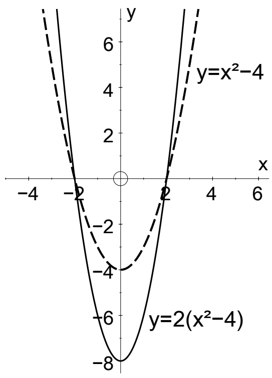



**In general** The equation of the graph of $y = x^2$, or $y = f(x)$, becomes $y = (x - a)^2 + b$, or $y = f(x - a) + b$, after a translation through $\binom{a}{b}$.

**Stretches**

(i) If the graph of $y = f(x)$ is stretched by a factor of $+2$ in the $y$-direction then the new equation is $y = 2 \times f(x)$.

*Example:*




The graph of $y = f(x) = x^2 - 4$ becomes

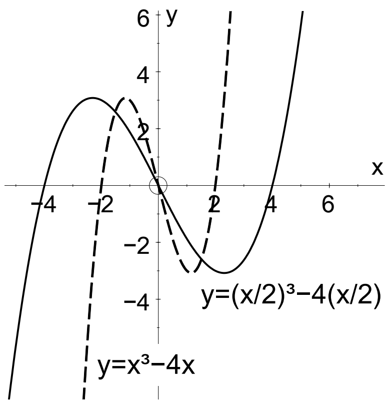
$y = 2f(x) = 2(x^2 - 4)$ after a stretch of factor 2 in the $y$-direction

In general $y = x^2$ or $y = f(x)$ becomes $y = ax^2$ or $y = af(x)$ after a stretch in the $y$-direction of factor $a$.

(ii) If the graph of $y = f(x)$ is stretched by a factor of $+2$ in the $x$-direction then the new equation is $y = f\left(\frac{x}{2}\right)$.

We replace $x$ by $\left(\frac{x}{2}\right)$ everywhere in the formula for $y$.

*Example:*




In this example the graph of

$y = x^3 - 4x$

has been stretched by a factor of 2 in the $x$-direction to form a new graph with equation

$y = f(x) \rightarrow y = f\left(\frac{x}{2}\right)$

$\Rightarrow \quad y = \left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)$

(iii) Note that to stretch by a factor of $\frac{1}{2}$ in the $x$-direction we replace $x$ by $\frac{x}{\frac{1}{2}} = 2x$ so that $y = f(x)$ becomes $y = f(2x)$

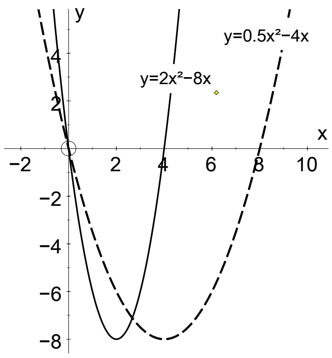
*Example:*

In this example the graph of $y = 0.5x^2 - 4x$

has been stretched by a factor of $\frac{1}{2}$ in the $x$-direction to form a new graph with equation

$$y = f(x) \quad \rightarrow \quad y = f(2x)$$

$$\Rightarrow \quad y = 0.5 \times \left(\frac{x}{\frac{1}{2}}\right)^2 - 4 \times \left(\frac{x}{\frac{1}{2}}\right) = 2x^2 - 8x$$




The new equation is formed by replacing $x$ by $\frac{x}{\frac{1}{2}} = 2x$ in the original equation.

**Reflections in the *x*-axis**

When reflecting in the $x$-axis all the positive $y$-coordinates become negative and vice versa

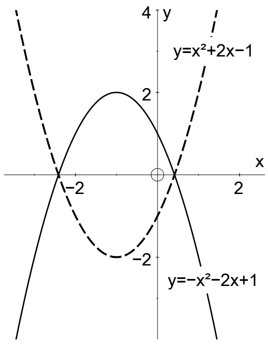
$\Rightarrow$ the image of $y = f(x)$ after reflection in the $x$-axis is $y = -f(x)$.

*Example:* The image of $y = f(x) = x^2 + 2x - 1$ after reflection in the $x$-axis is

$$y = f(x) \quad \rightarrow \quad y = -f(x)$$

$$\Rightarrow \quad y = -(x^2 + 2x - 1)$$

$$\Rightarrow \quad y = -x^2 - 2x + 1$$




**Reflections in the *y*–axis**

When reflecting in the $y$-axis

the $y$-coordinate for $x = +3$ becomes the $y$-coordinate for $x = -3$ and
the $y$-coordinate for $x = -2$ becomes the $y$-coordinate for $x = +2$.

Thus the equation of the new graph is found by replacing $x$ by $-x$

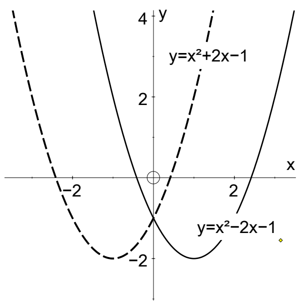
$\Rightarrow$ the image of $y = f(x)$ after reflection in the $y$–axis is $y = f(-x)$.

*Example:* The image of $y = f(x) = x^2 + 2x - 1$ after reflection in the $x$-axis is

$$y = f(x) \quad \rightarrow \quad y = f(-x)$$

$$\Rightarrow \quad y = (-x)^2 + 2(-x) - 1$$

$$\Rightarrow \quad y = x^2 - 2x - 1$$



**Summary of transformations**

| Old equation | Transformation | New equation |
|---|---|---|
| $y = f(x)$ | Translation through $\begin{pmatrix} a \\ b \end{pmatrix}$ | $y = f(x - a) + b$ |
| $y = f(x)$ | Stretch with factor $a$ in the $y$-direction. | $y = a \times f(x)$ |
| $y = f(x)$ | Stretch with factor $a$ in the $x$-direction. | $y = f\left(\frac{x}{a}\right)$ |
| $y = f(x)$ | Enlargement with factor $a$ centre (0, 0) | $y = a \times f\left(\frac{x}{a}\right)$ |
| $y = f(x)$ | Stretch with factor $\frac{1}{a}$ in the $x$-direction. | $y = f(ax)$ |
| $y = f(x)$ | Reflection in the $x$-axis | $y = -f(x)$ |
| $y = f(x)$ | Reflection in the $y$-axis | $y = f(-x)$ |

# 6 Differentiation

## General result

Differentiating is finding the gradient of the curve.

$y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$, or $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

*Examples:*

*a*) $y = 3x^2 - 7x + 4$, $\dfrac{dy}{dx} = 6x - 7$

*b*) $f(x) = 7\sqrt{x} = 7x^{1/2}$ $f'(x) = 7 \times \frac{1}{2} x^{\frac{-1}{2}} = \dfrac{7}{2\sqrt{x}}$

*c*) $y = \dfrac{8}{x^3} = 8x^{-3}$ $\dfrac{dy}{dx} = 8 \times (-3x^{-4}) = \dfrac{-24}{x^4}$

*d*) $f(x) = (2x + 1)(x - 3)$
multiply out first
$= 2x^2 - 5x - 3$ $f'(x) = 4x - 5$

*e*) $y = \dfrac{3x^7 - 4x^2}{x^5}$

split up first

$= \dfrac{3x^7}{x^5} - \dfrac{4x^2}{x^5} = 3x^2 - 4x^{-3}$ $\dfrac{dy}{dx} = 6x - (-12x^{-4}) = 6x + \dfrac{12}{x^4}$

## Tangents and Normals

**Tangents**

*Example:* Find the equation of the tangent to $y = 3x^2 - 7x + 5$ at the point where $x = 2$.

*Solution:*

We first find the gradient when $x = 2$.

$y = 3x^2 - 7x + 5 \Rightarrow \dfrac{dy}{dx} = 6x - 7$ and when $x = 2$, $\dfrac{dy}{dx} = 6 \times 2 - 7 = 5$.

so the gradient when $x = 2$ is 5. The equation is of the form $y = mx + c$ where $m$ is the gradient so we have

$$y = 5x + c.$$

To find $c$ we must find a point on the line, namely the point on the curve when $x = 2$.

When $x = 2, y = 3 \times 2^2 - 7 \times 2 + 5 = 3$.

$\Rightarrow$ the equation of the tangent is $y - 3 = 5(x - 2)$ using $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y = 5x - 7$

**Normals**

(The normal to a curve is the line which is perpendicular to the tangent at that point).

We first remember that if two lines with gradients $m_1$ and $m_2$ are perpendicular then $m_1 \times m_2 = -1$.

*Example:*

Find the equation of the normal to the curve $y = x + \frac{2}{x}$ at the point where $x = 2$.

*Solution:*

We first find the gradient of the tangent when $x = 2$.

$$y = x + 2x^{-1} \Rightarrow \frac{dy}{dx} = 1 - 2x^{-2} = 1 - \frac{2}{x^2}$$

$\Rightarrow$ when $x = 2$ gradient of the tangent is $m_T = 1 - \frac{2}{4} = \frac{1}{2}$.

If gradient of the normal is $m_N$ then $m_T \times m_N = -1 \Rightarrow \frac{1}{2} \times m_N = -1 \Rightarrow m_N = -2$

When $x = 2$, $y = 2 + \frac{2}{2} = 3$

$\Rightarrow$ the equation of the normal is $y - 3 = -2(x - 2)$ using $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y = -2x + 7$.

# 7 Integration

## *Indefinite integrals*

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{provided that } n \neq -1$$

**N.B. NEVER FORGET THE ARBITRARY CONSTANT + *C*.**

*Examples:*

*a*) $\displaystyle\int \frac{4}{3x^6} \, dx = \int \frac{4x^{-6}}{3} \, dx$

$= \frac{4}{3} \times \frac{x^{-5}}{-5} + C = \frac{-4}{15x^5} + C.$

*b*) $\displaystyle\int (3x-2)(x+1) \, dx = \int 3x^2 + x - 2 \, dx$ multiply out first

$= x^3 + \frac{1}{2}x^2 - 2x + C.$

*c*) $\displaystyle\int \frac{x^9 + 5x^2}{x^5} \, dx = \int \frac{x^9}{x^5} + \frac{5x^2}{x^5} \, dx$ split up first

$= \displaystyle\int x^4 + 5x^{-3} \, dx$

$= \displaystyle\frac{x^5}{5} + 5\frac{x^{-2}}{-2} + C = \frac{x^5}{5} - \frac{5}{2x^2} + C.$

**Finding the arbitrary constant**

If you know $\frac{dy}{dx}$ and the coordinates of a point on the curve you can find the arbitrary constant, *C*.

*Example:* Solve $\frac{dy}{dx} = 3x^2 - 5$, given that the curve passes through the point (2, 4).

*Solution:* $y = \int 3x^2 - 5 \, dx = x^3 - 5x + C$

$y = 4$ when $x = 2$

$\Rightarrow \quad 4 = 2^3 - 5 \times 2 + C \quad \Rightarrow \quad C = 6$

$\Rightarrow \quad y = x^3 - 5x + 6.$

# Appendix

## Quadratic equation formula proof

$ax^2 + bx + c = 0$

$\Rightarrow \quad ax^2 + bx = -c$

$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$ — ÷ by $a$ to make coefficient of $x^2 = +1$

$\Rightarrow \quad x^2 + \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ — add (half coefficient of $x$)$^2$ to both sides

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ — complete square

$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$ — square root, do not forget ±

$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$

## Differentiation from first principles

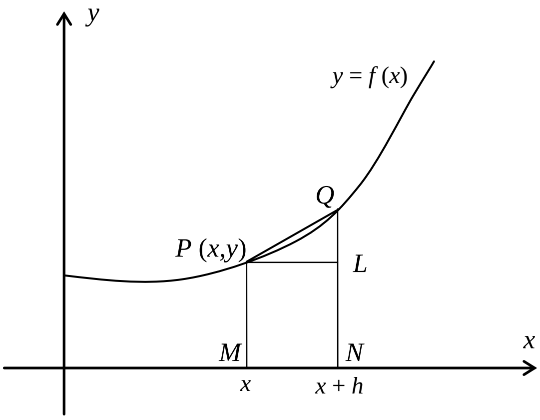
If $P$ and $Q$ are close together, the gradient of $PQ$ will be nearly equal to the gradient of the tangent of the curve at $P$.

Let the $x$-coordinates of $M$ be $x$, and of $N$ be $(x + h)$.

$\Rightarrow \quad PM = f(x)$ and $QN = f(x + h)$

$\Rightarrow \quad QL = QN - PM = f(x + h) - f(x)$

and $PL = h$.




The gradient of $PQ = \frac{\text{increase in } y}{\text{increase in } x} = \frac{\delta y}{\delta x} = \frac{QL}{PL} = \frac{f(x+h) - f(x)}{h}$

As $h \to 0$, $Q$ gets closer and closer to $P$, and the gradient of $PQ$ gets closer and closer to the gradient of the curve at $P$.

We write $f'(x)$ to mean the gradient of the curve (or tangent) at $P$,

$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

and we also write the gradient as $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

*Example:* Find, from first principles, the gradient of $f(x) = x^3$.

*Solution:* $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 \qquad \text{cancelling } h$$

$$\Rightarrow \quad f'(x) = 3x^2 + 0 + 0 \qquad \text{putting } h = 0 \text{ (which we can do after cancelling } h\text{)}$$

$$\Rightarrow \quad f'(x) = 3x^2, \quad \text{or} \quad \frac{dy}{dx} = 3x^2.$$

*Example:* Find, from first principles, the gradient of $f(x) = \frac{5}{x}$.

*Solution:* $f(x) = \frac{5}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} = \lim_{h \to 0} \frac{5x - 5(x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-5h}{hx(x+h)} = \lim_{h \to 0} \frac{-5}{x(x+h)} \qquad \text{cancelling } h$$

$$\Rightarrow \quad f'(x) = \frac{-5}{x(x+0)} \qquad \text{putting } h = 0 \text{ (which we can do after cancelling } h\text{)}$$

$$\Rightarrow \quad f'(x) = \frac{-5}{x^2}, \quad \text{or} \quad \frac{dy}{dx} = \frac{-5}{x^2}.$$

**General formulae**

For any function $f(x)$, the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

The second formula comes from taking two points, $Q$ and $Q'$, equally placed on either side of $P$. $Q'$ has $x$-coord $x - h$, and $Q$ has $x$-coord $x + h$.

# Index